

Categories, Homotopy  
and  
0-minimality

This talk is a glimpse into this paper:

## MODEL CATEGORIES FOR O-MINIMAL GEOMETRY

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ABSTRACT. We introduce a model category of spaces based on the definable sets of an o-minimal expansion of a real closed field. As a model category, it resembles the category of topological spaces, but its underlying category is a coherent topos. We will show in future work that its cofibrant objects are precisely the “weak polytopes” of Knebusch.

\* The most relevant sections are 1-4.

# Plan

- Motivations
- $\mathcal{O}$ -minimality & Tameness
- Categories  $\mathbb{D}$ ,  $\mathbb{D}$ ,  $\mathbb{P}$
- Towards the model category

Motivations

Two models for the homotopy theory of spaces:

Simplicial Sets

Algebraically nice

Topological spaces

Geometrically natural

a middle ground

# The main theorem:

**1.1. Theorem** (Theorems 11.10 and 11.30). *Fix an o-minimal structure and let  $\mathbb{P}$  denote the category of closed and bounded definable sets and continuous definable functions, equipped with the Grothendieck topology generated by finite jointly surjective families. Then the geometric realization–Sing adjunction*

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Shv}(\mathbb{P}) : \mathbf{Sing}$$

*induces a model category structure on  $\mathbf{Shv}(\mathbb{P})$  in which every object is fibrant. Furthermore, this adjunction becomes a Quillen equivalence.*

$O$ -minimality  $\times$  Tameness

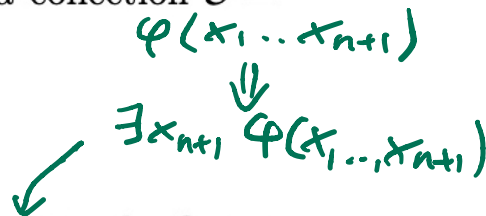
## Structures :

2.1. **Definition** ([18, 1.2.1]). A *structure* on a set  $R$  is a collection  $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$  that satisfies the following properties:

- (i)  $\mathcal{S}_n$  is a boolean subalgebra of the subsets of  $R^n$ .
- (ii) If  $X \in \mathcal{S}_n$ , then  $R \times X$  and  $X \times R$  are in  $\mathcal{S}_{n+1}$ .
- (iii) The set  $\{(x_1, \dots, x_n) \mid x_1 = x_n\}$  is in  $\mathcal{S}_n$ .
- (iv) If  $X \in \mathcal{S}_{n+1}$ , and  $\pi : R^{n+1} \rightarrow R^n$  is the projection onto the first  $n$  coordinates, then  $\pi(X)$  is in  $\mathcal{S}_n$ .

A set  $X \subseteq R^n$  is called  $\mathcal{S}$ -*definable* or *definable in  $\mathcal{S}$*  if  $X \in \mathcal{S}_n$ . A function  $f : X \rightarrow Y$  for  $X \subseteq R^m$  and  $Y \subseteq R^n$  is  $\mathcal{S}$ -*definable* if its graph is  $\mathcal{S}$ -definable. Usually the structure  $\mathcal{S}$  will be clear from the context, and we simply speak of *definable* sets and functions.

- The image or preimage of a definable set under a definable function is definable.
- The composition of definable functions is definable.
- ...



## Signature:

2.4. **Remark.** Let  $\mathcal{S}$  be a structure on  $R$ . Consider the signature  $L_{\mathcal{S}}$  with, for each  $n \geq 0$ , one  $n$ -ary relation symbol corresponding to each definable subset  $X \subseteq R^n$ , and one  $n$ -ary function symbol corresponding to each definable function  $f : R^n \rightarrow R$ . The set  $R$  becomes a structure for  $L_{\mathcal{S}}$  (in the model-theoretic sense) in a tautological way. Then, for any first-order formula  $\varphi$  in  $n$  free variables over the signature  $L_{\mathcal{S}}$ , the subset of  $R^n$  that  $\varphi$  cuts out is again definable.



definability is closed under FOL

## $\mathcal{O}$ -minimal structures:

2.6. **Definition** ([18, 1.3.2]). Let  $R$  be a set equipped with a dense linear order  $<$  without endpoints, and let  $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$  be a structure on  $R$ . Then  $\mathcal{S}$  is *o-minimal* if it satisfies the following two properties:

- (i) The set  $\{(x, y) \mid x < y\}$  is in  $\mathcal{S}_2$ .
- (ii)  $\mathcal{S}_1$  consists exactly of those sets that are finite unions of points and (possibly unbounded) open intervals in  $R^1 = R$ .

In this paper we assume, in addition to these basic axioms, that  $R$  is an ordered ring and that the sets

$$\{(x, y, z) \mid x + y = z\} \quad \text{and} \quad \{(x, y, z) \mid x \cdot y = z\}$$

are definable. In this case  $R$  must be a real closed field with its induced ordering [18, Proposition 1.4.6]. We say that  $\mathcal{S}$  *extends* the real closed field  $R$ .

think of it as  $\mathbb{R}$

(with the exception of  $\tilde{\mathbb{Q}}$  later)

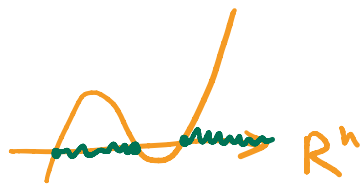
Examples:

\*  $(\mathbb{R}, <, +, \cdot)$

\*  $(\tilde{\mathbb{Q}}, <, +, \cdot)$

\*  $R_{sa}$

Structure formed by semialgebraic sets



no quantifiers

Non-examples:

\*  $(\mathbb{Z}, <, +, \cdot)$

\*  $(\mathbb{Q}, <, +, \cdot)$

\*  $(\mathbb{R}, <, +, \cdot, \sin(\cdot))$

↳ those that belong to the boolean algebra generated by sets of the form

$\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$  with  $f \in \mathbb{R}[X_1, \dots, X_n]$

Example:

$\mathcal{R}_{SA}$  is indeed  $\mathcal{O}$ -minimal

① BA

②  $\mathcal{R} \times X$  &  $X \times \mathcal{R}$

③  $\{(x_1, \dots, x_n) \mid x_i = x_n\}$

★ ④  $\pi(X)$

⑤  $\{(x, y) \mid x < y\} \in S_2$

⑥ finite union of points & intervals for  $S_1$

Tarski-Seidenberg theorem

↑ quantifier-elimination  
for real closed fields



$\forall (x_1, \dots, x_n)$

$\Leftrightarrow \exists x^{n+1} \varphi(x_1, \dots, x_{n+1})$

→ polynomials in one variable have finitely many roots (except functions like  $f(x) = x^0 - 1$ )

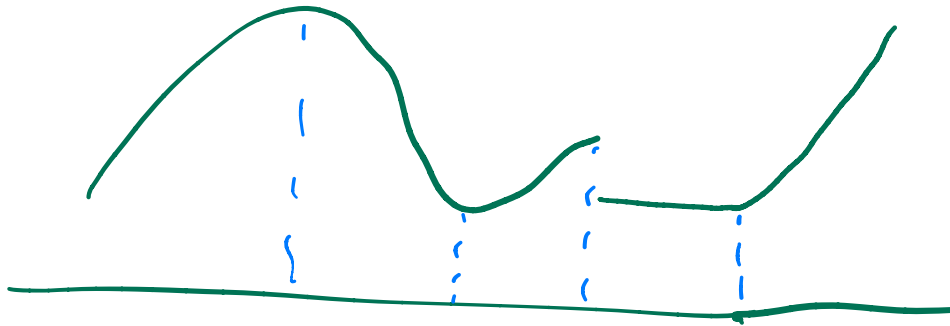
Some tameness properties:

\* Monotonicity Theorem

\* Uniform Finiteness

\* Cell Decomposition Theorem

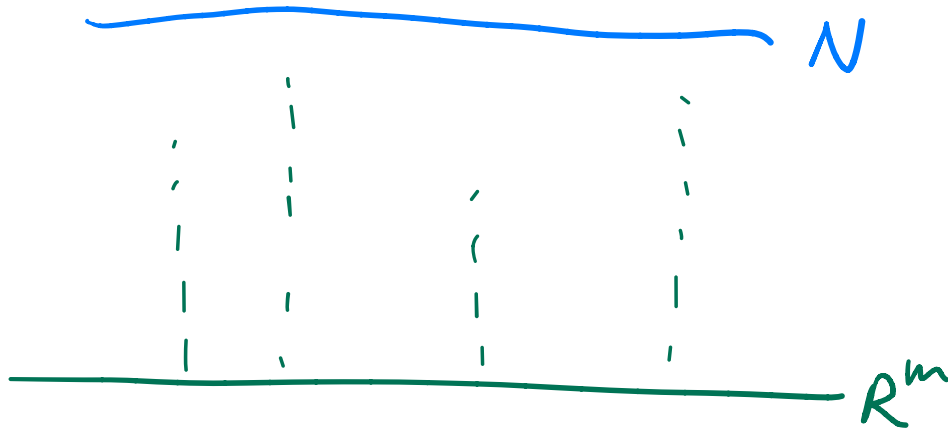
**(1.2) MONOTONICITY THEOREM.** *Let  $f:(a,b) \rightarrow \mathbb{R}$  be a definable function on the interval  $(a,b)$ . Then there are points  $a_1 < \dots < a_k$  in  $(a,b)$  such that on each subinterval  $(a_j, a_{j+1})$ , with  $a_0 = a$ ,  $a_{k+1} = b$ , the function is either constant, or strictly monotone and continuous.*



(2.13) LEMMA (UNIFORM FINITENESS PROPERTY). Suppose the definable subset  $Y$  of  $R^{m+1}$  is finite over  $R^m$ . Then  $Y$  is uniformly finite over  $R^m$ .

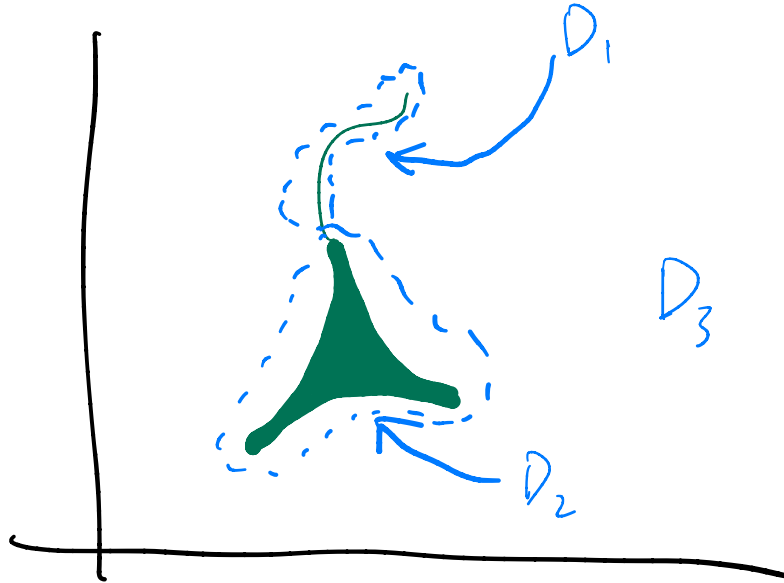
$Y \subseteq R^{m+1}$  is finite if for each  $x \in R^m$   
the fibre  $Y_x$  is finite

$\exists N \in \mathbb{N}$  s.t.  $|Y_x| < N$   
for all  $x \in R^m$



(2.11) CELL DECOMPOSITION THEOREM.

- (I<sub>m</sub>) Given any definable sets  $A_1, \dots, A_k \subseteq R^m$  there is a decomposition of  $R^m$  partitioning each of  $A_1, \dots, A_k$ .
- (II<sub>m</sub>) For each definable function  $f: A \rightarrow R$ ,  $A \subseteq R^m$ , there is a decomposition  $\mathcal{D}$  of  $R^m$  partitioning  $A$  such that the restriction  $f|_B: B \rightarrow R$  to each cell  $B \in \mathcal{D}$  with  $B \subseteq A$  is continuous.



## Continuity:

2.12. **Lemma.** Let  $f : X \rightarrow Y$  be a definable function between definable sets  $X \subseteq \mathbb{R}^m$  and  $Y \subseteq \mathbb{R}^n$ . Then  $f$  is continuous if and only if  $f^{-1}(U)$  is open in  $X$  for every definable open subset  $U$  of  $Y$ . Similarly, we can also test continuity using definable closed subsets.

## Closure:

2.13. **Example.** If  $X \subseteq \mathbb{R}^n$  is definable then its closure  $\overline{X}$  is also definable, using [Remark 2.4](#), since

$$(y_1, \dots, y_n) \in \overline{X} \iff \forall \varepsilon > 0, \exists (x_1, \dots, x_n) \in X, |x_1 - y_1| < \varepsilon \wedge \dots \wedge |x_n - y_n| < \varepsilon.$$

\* homeomorphism, embedding, etc. also apply unchanged.

But we need to modify the definitions for

## \* Connectedness

2.15. **Definition** ([18, Definition 1.3.5]). A definable set is *definably connected* if it is nonempty and cannot be written as the disjoint union of two nonempty definable open subsets.

## \* Compactness

2.16. **Definition.** A subset  $X \subseteq R^n$  is called *bounded* if there exists an  $r \in R$  such that  $X \subseteq [-r, r]^n$ . It is *definably compact* if it is definable, closed and bounded. (There is also an important relative version of this notion, *properness*, which we discuss in [Section 5](#).)

e.g.  $\tilde{\mathbb{Q}}$  is not Dedekind complete

Categories  $\mathbb{D}$ ,  $\mathbb{D}$ ,  $\mathbb{P}$

# Morphisms in $\mathcal{D}$ :

3.1. **Definition.** Let  $R$  and  $R'$  be two real closed fields, and let  $\mathcal{S}$  and  $\mathcal{S}'$  be  $\mathcal{o}$ -minimal structures expanding  $R$  and  $R'$  respectively. A *morphism of  $\mathcal{o}$ -minimal structures* from  $\mathcal{S}$  to  $\mathcal{S}'$  is a pair  $(\phi, \Phi)$  consisting of a homomorphism of ordered rings  $\phi : R \rightarrow R'$  and a collection of functions  $\Phi = (\Phi_n : \mathcal{S}_n \rightarrow \mathcal{S}'_n)_{n \in \mathbb{N}}$  satisfying the following conditions:

- (i) The functions  $\Phi_n$  commute with the operations under which the definable sets of a structure are closed as described in [Definition 2.1](#). Namely, the following equations hold whenever they make sense:

$$\Phi_n(\emptyset) = \emptyset,$$

$$\Phi_n(X \cup Y) = \Phi_n(X) \cup \Phi_n(Y),$$

$$\Phi_n(R^n - X) = R'^n - \Phi_n(X),$$

$$\Phi_{n+1}(R \times X) = R' \times \Phi_n(X),$$

$$\Phi_{n+1}(X \times R) = \Phi_n(X) \times R',$$

$$\Phi_n(\{(x_1, \dots, x_n) \in R^n \mid x_1 = x_n\}) = \{(x_1, \dots, x_n) \in R'^n \mid x_1 = x_n\},$$

$$\Phi_n(\pi_R(X)) = \pi_{R'}(\Phi_{n+1}(X))$$

where in the last equation,  $\pi_R : R^{n+1} \rightarrow R^n$  and  $\pi_{R'} : R'^{n+1} \rightarrow R'^n$  are the projections onto the first  $n$  coordinates.

- (ii) For every  $r \in R$ , we have  $\Phi_1(\{r\}) = \{\phi(r)\}$ .  
(iii)  $\Phi_2(\{(x, y) \in R^2 \mid x < y\}) = \{(x, y) \in R'^2 \mid x < y\}$ .  
(iv) Likewise,  $\Phi_3$  takes the graphs of addition and multiplication on  $R$  to the graphs of the corresponding operations on  $R'$ .

Because of condition (ii), the homomorphism  $\phi$  is completely determined by  $\Phi_1$ . For this reason, we will abuse notation and also write  $\Phi$  for this induced function  $R \rightarrow R'$ , as well as for the entire morphism of  $\mathcal{o}$ -minimal structures. We also suppress the subscripts on the individual operations  $\Phi_n$ .

Can we  
simply require

$$\Phi_1(\{1_R\}) = \{1_{R'}\} ?$$

Examples:

\*  $S \subseteq S'$

$$(R, S) \xrightarrow{(\text{id}, \Phi)} (R, S')$$

$\Phi(X) = X$

\*  $R \subseteq R'$

$$\Phi: R_{sa} \rightarrow R'_{sa}$$

s.t.  $\Phi(X) = \llbracket \psi \rrbracket^{R'}$  where  $\llbracket \psi \rrbracket^R = X$ .

well-defined as the theory of real closed field  
is model-complete  
 $\Rightarrow$  every embedding is an elementary embedding

The initial object:

3.3. **Lemma.** *The o-minimal structure  $\tilde{\mathcal{Q}}_{\text{sa}}$  of semialgebraic sets over the algebraic real numbers is initial in the category of o-minimal structures with morphisms as defined in Definition 3.1.*

$\tilde{\mathcal{Q}}$  is the initial  
real closed field



+

semialgebraic structure  
is the smallest possible structure

3.4. **Example.** Let  $\Phi : (R, \mathcal{S}) \rightarrow (R', \mathcal{S}')$  be any morphism of o-minimal structures. Then

$$\Phi(\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \leq 1\}) = \{(x, y) \in \mathbb{R}'^2 \mid 0 \leq x \leq y \leq 1\}$$



a ring homomorphism

$$\Phi(0) = 0$$

$$\Phi(1) = 1$$

Note: In cases like  $\widehat{\mathbb{Q}} \subset \mathbb{R}$ ,  $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \leq 1\}$

contains points not of the form  $(\Phi(x), \Phi(y))$ . So in general,

$$\Phi(X) \neq \{\Phi(x) : x \in X\}.$$

But when  $X$  is finite, this holds.

3.5. Fix  $r, n_1, \dots, n_r \geq 0$  and let  $P$  be a predicate of  $r$  definable sets  $X_1 \subseteq R^{n_1}, \dots, X_r \subseteq R^{n_r}$  in any o-minimal structure  $(R, \mathcal{S})$  with the following property: there exists a single sentence  $\psi$  in the language of an ordered ring augmented by relation symbols of arity  $n_1, \dots, n_r$ , such that  $P(X_1, \dots, X_r)$  holds if and only if  $\psi$  is valid in  $R$  when the  $i$ th relation symbol is interpreted as  $X_i \subseteq R^{n_i}$ . For example,

- with  $r = 1$ ,  $P(X)$  might be the predicate “ $X$  is nonempty” or the predicate “ $X \subseteq R^{n_1}$  is closed”;



Suppose  $X \subseteq R$ , we can introduce a new relation symbol  $Q$ .

Let  $\gamma := \exists x, Qx$ .

interpreted as  $x \in X$ .

We refer to such predicates as *first-order properties*.

Let  $\Phi : (R, \mathcal{S}) \rightarrow (R', \mathcal{S}')$  be a morphism of o-minimal structures, and let  $X_1 \subseteq R^{n_1}, \dots, X_r \subseteq R^{n_r}$  be  $\mathcal{S}$ -definable sets. Then we have the following “transfer principle”:

- (\*) For any first-order property  $P$ ,  $P(\Phi(X_1), \dots, \Phi(X_r))$  holds if and only if  $P(X_1, \dots, X_r)$  does.

A property is FO-definable



invariant under morphisms

# the transfer principle



**3.6. Lemma.** Let  $\Phi : (R, \mathcal{S}) \rightarrow (R', \mathcal{S}')$  be a morphism of o-minimal structures.

- (i) For every  $\mathcal{S}$ -definable subset  $X$ , we have  $\Phi(X) = \emptyset \iff X = \emptyset$ .
- (ii) For all  $\mathcal{S}$ -definable subsets  $X$  and  $Y$ , we have  $\Phi(X) \subseteq \Phi(Y) \iff X \subseteq Y$ .
- (iii) If  $f : X \rightarrow Y$  is a definable function with graph  $\Gamma_f$ , then  $\Phi(\Gamma_f)$  is the graph of a definable function from  $\Phi(X)$  to  $\Phi(Y)$ .

We denote this function by  $\Phi f : \Phi X \rightarrow \Phi Y$ . In the remaining items, all functions are assumed to be definable functions between definable sets.

- (iv) If  $f : X \rightarrow Y$  is **continuous** then  $\Phi f : \Phi X \rightarrow \Phi Y$  is continuous.
- (v) If  $X$  is closed and bounded then  $\Phi X$  is closed and bounded.
- (vi) If  $f : X \rightarrow Y$  is injective then  $\Phi f$  is injective.
- (vii) If  $f : X \rightarrow Y$  is a **homeomorphism** then  $\Phi f$  is a homeomorphism.
- (viii) If  $f : X \rightarrow Y$  is an **embedding** then  $\Phi f$  is an embedding.
- (ix) If  $f_1 : X_1 \rightarrow Y, \dots, f_n : X_n \rightarrow Y$  are jointly surjective, then  $\Phi f_1, \dots, \Phi f_n$  are jointly surjective.
- (x)  $\Phi$  preserves identities and composition.

# Categories $\mathbb{D}$ & $\mathbb{P}$

4.1. **Definition.** (i) We write  $\mathbb{D} = \mathbb{D}(R, \mathcal{S})$  for the category of definable sets. More precisely, an object of  $\mathbb{D}$  is a pair  $(n, X)$  where  $n \geq 0$  and  $X$  is a definable set in  $R^n$ . A morphism of  $\mathbb{D}$  is a continuous definable function.

(ii) We write  $\mathbb{P} = \mathbb{P}(R, \mathcal{S})$  for the full subcategory of  $\mathbb{D}$  on the definable sets which are definably compact (i.e., closed and bounded). We refer to the objects of  $\mathbb{P}$  as *polytopes*.

The property of being definably compact is invariant under definable homeomorphism

$\Downarrow$   
 $\mathbb{P}$  is a replete subcategory of  $\mathbb{D}$

## $\Phi$ as a functor

4.6. Let  $\Phi : (R, \mathcal{S}) \rightarrow (R', \mathcal{S}')$  be a morphism of o-minimal structures. Then there is a functor

$$\begin{aligned} \mathbb{D}(R, \mathcal{S}) &\rightarrow \mathbb{D}(R', \mathcal{S}') \\ X &\mapsto \Phi(X) \\ f &\mapsto \Phi(f); \end{aligned}$$

see [Lemma 3.6](#) for more on  $\Phi(f)$ . This functor restricts to a functor  $\mathbb{P}(R, \mathcal{S}) \rightarrow \mathbb{P}(R', \mathcal{S}')$ , since  $\Phi(X)$  is a polytope if  $X$  is. We call these functors the *associated functors* of  $\Phi$ , and denote them also by  $\Phi$ .

# Limits & Colimits

$\Delta$  &  $\mathbb{P}$

Finite limits

✓

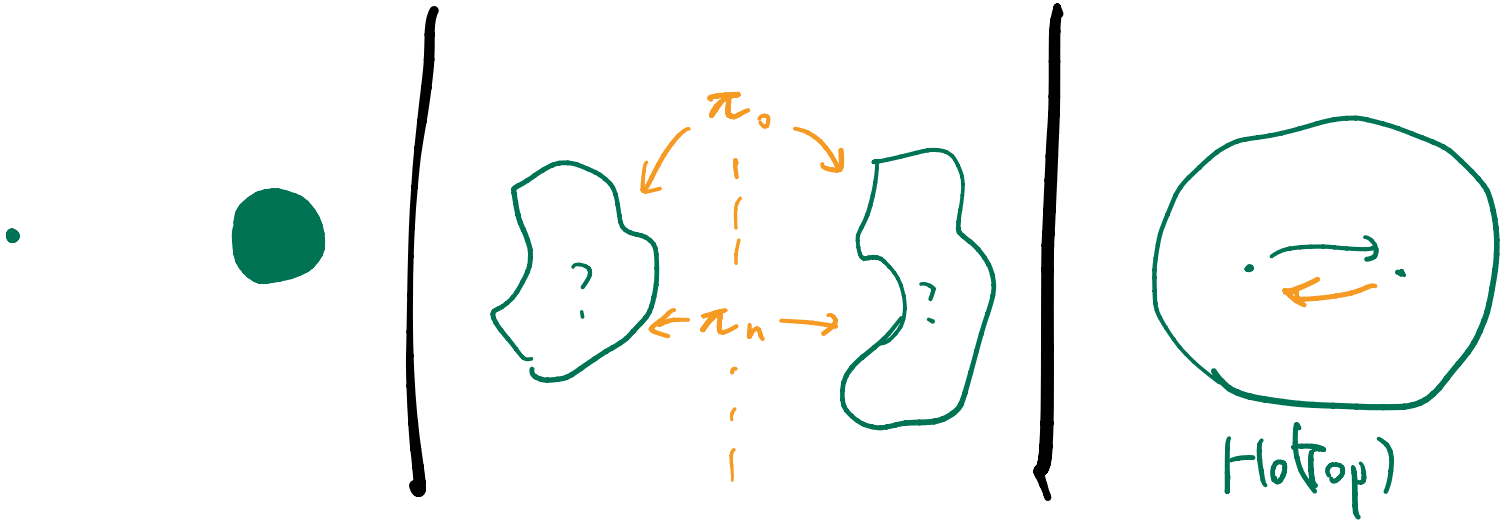
Finite colimits

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# Towards the model category

(an informal intro)

# Homotopy, Weak equivalences & Ho(Top)



# Model Categories

A model category is a category equipped with a model structure  $(W, C, F)$ .

weak equivalences

cofibrations

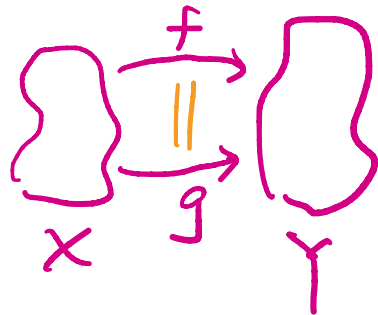
fibrations

build cylinder objects like

$$X \times I$$

build path objects like

$$Y^I$$



# Quillen adjunctions & Quillen equivalences

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathbf{Sing}$$



$$Ho(\mathbf{sSet}) \simeq Ho(\mathbf{Top})$$

# Towards the model category

$$|-| : \mathbf{sSet} \rightleftarrows ? : \mathbf{Sing}$$

$$\mathbb{P} \xrightarrow{\text{free co completion}} \mathbf{Psh}(\mathbb{P}) \xrightarrow{\text{correct glueing}} \mathbf{Shv}(\mathbb{P})$$

1.1. **Theorem** (Theorems 11.10 and 11.30). *Fix an o-minimal structure and let  $\mathbb{P}$  denote the category of closed and bounded definable sets and continuous definable functions, equipped with the Grothendieck topology generated by finite jointly surjective families. Then the geometric realization–Sing adjunction*

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*induces a model category structure on  $\mathbf{Shv}(\mathbb{P})$  in which every object is fibrant. Furthermore, this adjunction becomes a Quillen equivalence.*

# Summary

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- $\mathcal{O}$ -minimality & Tameness
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# Main Reference

Tame Topology and  $\mathcal{O}$ -minimal Structures

Model Categories for  $\mathcal{O}$ -minimal Geometry