Categories for DEL

2nd part

Goal:

- Recast the semantics of DEL in categorical terms.

  public announcements ~ submodels

  product updates ~ Products

Benefits:

- Reveal the structural unity behind

  (the notion of common knowledge, product update, etc. is not an arbitrary or ad hoc construction since they are categorically canonical in some sense)

- Smooth integration with other logic

  (e.g. FO-DEL)

One of the most important facts:

Kr is topological over Sets !

Fact 3. **Kr** is "topological over **Sets**",[15] meaning, concretely, the following. Given any family of functions $f_i : X \to Y_i$ $(i \in I)$ to Kripke frames (Y_i, R_i), the relation

$$wR_X v \iff f_i(w) R_i f_i(v) \text{ for all } i \in I, \quad \text{i.e.} \quad R_X = \bigcap_{i \in I} (f_i^{\dagger} \circ R_i \circ f_i),$$

is the (unique) "initial lift" of $\{f_i\}_{i \in I}$, i.e. the relation on X such that, given any function $g : Z \to X$, all $f_i \circ g$ are monotone from a frame (Z, R_Z) iff g is.

(In fact, Fact 3 holds of **Kr** in general, again with R^{α} in place of R.) One may note that the relation

Questions

&

Recap

Questions & A quick recap

- Do we have to mention the duality results $\mathbf{Rel} \sim \mathrm{CABAs}$, $\mathbf{Kr} \sim \mathrm{CABAOs}$?
- What do we need $\mathbf{Rel}$ for? Why don't we jump straight into $\mathbf{Kr}$?
- What do we need tabulations for?
- Why do we need monotone but not bounded morphisms for dynamic updates?

- Do we have to mention the duality results Rel ~ CABAs, Kr ~ CABAOs ?

Not really. It is not techincally necessary but gives us a bigger picture of the categorical/algebraic duality theory.

- What do we need Rel for? Why don't we jump straight into Kr?

this generalization gives us $\forall_{R'} t$, $\exists_{R'} t$ (see public announcement)

Rel

$$\begin{tikzcd} X \arrow[loop above, "R"] \arrow[r, "R'"] & Y \end{tikzcd}$$

Single & Static Models

(monotone/bounded)

Kr

$$(X, R_X) \xrightarrow{f} (Y, R_Y)$$

Multiple & Dynamic Models

(Model updates)

- What do we need tabulations for?

  In Kripke semantics, we use relations to interpret modal operators:

Kripke semantics uses binary relations to interpret unary modal operators. A Kripke frame is a set X paired with a binary relation $R : X \nrightarrow X$, and a Kripke model is a Kripke frame (X, R) equipped with an assignment $[\![-]\!]$ of subsets $[\![p]\!] \subseteq X$ to propositional variables p. In fact we extend the notation to all propositions φ, so that $w \in [\![\varphi]\!] \subseteq X$ means that φ is true at w. Now, given a relation $R : X \nrightarrow Y$, define two monotone maps $\exists_R, \forall_R : \mathscr{P}X \to \mathscr{P}Y$ by

$$\exists_R(S) = \{v \in Y \mid w \in S \text{ for some } w \in X \text{ such that } wRv\},$$
$$\forall_R(S) = \{v \in Y \mid w \in S \text{ for all } w \in X \text{ such that } wRv\}.$$

$\Diamond_R = \exists_{R^\dagger}$

$\Box_R = \forall_{R^\dagger}$

for $R : X \nrightarrow X$

Then, for a relation $R : X \nrightarrow X$ on a set X, $\exists_{R^\dagger}, \forall_{R^\dagger} : \mathscr{P}X \to \mathscr{P}X$ interpret the "possibility" operator $\Diamond$ and the "necessity" operator $\Box$, respectively—i.e.

$$[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!], \qquad [\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]. \tag{2}$$

But in Kr.

$$(X, R_X) \xrightarrow{R} (Y, R_Y)$$

R is not a morphism. But we may need to use R for defining dynamic operators as across-model modalities (see later).

$$\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y
\end{tikzcd}
\qquad R = r_2 \circ r_1^{\dagger}$$

But does such (R, R_R) exist?

$$\begin{tikzcd}
 & (R, R_R) \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
(X, R_X) & & (Y, R_Y)
\end{tikzcd}$$

Fact 3 gives us a canonical construction of R_R.

$$R = r_2 \circ r_1^\dagger$$

$$\forall_{R^\dagger} = \forall_{r_1} \circ r_2^{-1} \quad , \quad \exists_{R^\dagger} = \exists_{r_1} \circ r_2^{-1}$$

(See the event update example for a real use)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
X \otimes E = \sum_{e \in E} [\![Pre(e)]\!]_X \arrow[r, hook, "i"] \arrow[rr, bend left, "p_E"] \arrow[dr, shift left, "p_X"] & X \times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
[\![Pre(e)]\!]_X \arrow[u, tail, "q_e"] \arrow[r, hook, "i_e"'] \arrow[ur, phantom, very near start, "\urcorner"] & X \arrow[u, tail, shift left, "q_e'"] \arrow[ul, shift left, "R_e", "|" marking] &
\end{tikzcd}
$$

coproduct injections

$R_{X \otimes E}$ is the initial lift of p_X and p_E.

$$[\![[E,e]\varphi]\!]_X = \forall_{R_e^{\dagger}} [\![\varphi]\!]_{X \otimes E}$$

$$[\![\langle E,e \rangle \varphi]\!]_X = \exists_{R_e^{\dagger}} [\![\varphi]\!]_{X \otimes E}$$

Tabulation: $R_e = q_e \circ i_e^{\dagger}$

- Why do we need monotone but not bounded morphisms for dynamic updates?

Monotone:

$$f \circ R_X \subseteq R_Y \circ f$$

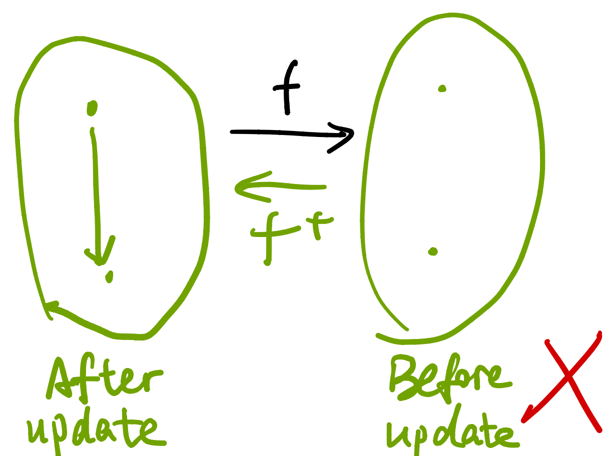
$(w R_X v \Rightarrow f(w) R_Y f(v))$

The agent cannot 'unknow'!

Bounded:

$$f \circ R_X = R_Y \circ f$$

(bisimulation)



Bounded morphisms imply that no event can teach agents anything (see the event update example later).

p_x is bounded

$\Rightarrow [\![\varphi]\!]_{X \otimes E} = p_x^{-1} [\![\varphi]\!]_X$ for every φ

$\Rightarrow [E, e]\varphi \equiv pre(e) \Rightarrow \varphi$ (including modal formulas!)

First part

Categories for DEL

2nd part

Goal:

- Recast the semantics of DEL in categorical terms.

public announcements ~ submodels
product updates ~ Products

Benefits:

- Reveal the structural unity behind
  (the notion of common knowledge, product update, etc. is not an arbitrary or ad hoc construction since they are categorically canonical in some sense)
- Smooth integration with other logic
  (e.g. FO-DEL)

- Categorical Look at ML

- Categorical look at DEL

- Application: combine FOL and DEL

A categorical look at Kripke semantics

The category of relations

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend left, "R", "|" marking] & Y \arrow[loop right, "1_Y"] \arrow[l, bend left, "R^\dagger", "|" marking] \end{tikzcd}$$

$$\boxed{\mathbf{Rel}}$$

Objects:

Sets

Morphisms:

Relations

$R \subseteq X \times Y$

Composition:

$$X \overset{R_1}{\nrightarrow} Y \overset{R_2}{\nrightarrow} Z$$

$$w \quad R_2 \circ R_1 \quad u$$

iff $\exists\ v \in Y$ s.t. $w R_1 v R_2 u$.

Examples:

- reflexivity of $R : X \nrightarrow X$

$$1_X \subseteq R$$

- transitivity of $R : X \nrightarrow X$

$$R \circ R \subseteq R$$

Facts:

- Rel is a dagger cat
- Rel is a higher cat
- Sets is a subcat of Rel
- $R \subseteq X \times Y$ can be tabulated in Sets

$$\begin{tikzcd} & R \subseteq X \times Y \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}$$

s.t. $R = r_2 \circ r_1^{\dagger}$

Relation–Modality duality

Kripke semantics uses binary relations to interpret unary modal operators. A Kripke frame is a set X paired with a binary relation $R : X \nrightarrow X$, and a Kripke model is a Kripke frame (X,R) equipped with an assignment $[\![-]\!]$ of subsets $[\![p]\!] \subseteq X$ to propositional variables p. In fact we extend the notation to all propositions φ, so that $w \in [\![\varphi]\!] \subseteq X$ means that φ is true at w. Now, given a relation $R : X \nrightarrow Y$, define two monotone maps $\exists_R, \forall_R : \mathscr{P}X \to \mathscr{P}Y$ by

$$\exists_R(S) = \{v \in Y \mid w \in S \text{ for some } w \in X \text{ such that } wRv\},$$
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$\Diamond_R = \exists_{R^\dagger}$

$\Box_R = \forall_{R^\dagger}$

for $R : X \nrightarrow X$

Then, for a relation $R : X \nrightarrow X$ on a set X, $\exists_{R^\dagger}, \forall_{R^\dagger} : \mathscr{P}X \to \mathscr{P}X$ interpret the "possibility" operator $\Diamond$ and the "necessity" operator $\Box$, respectively—i.e.

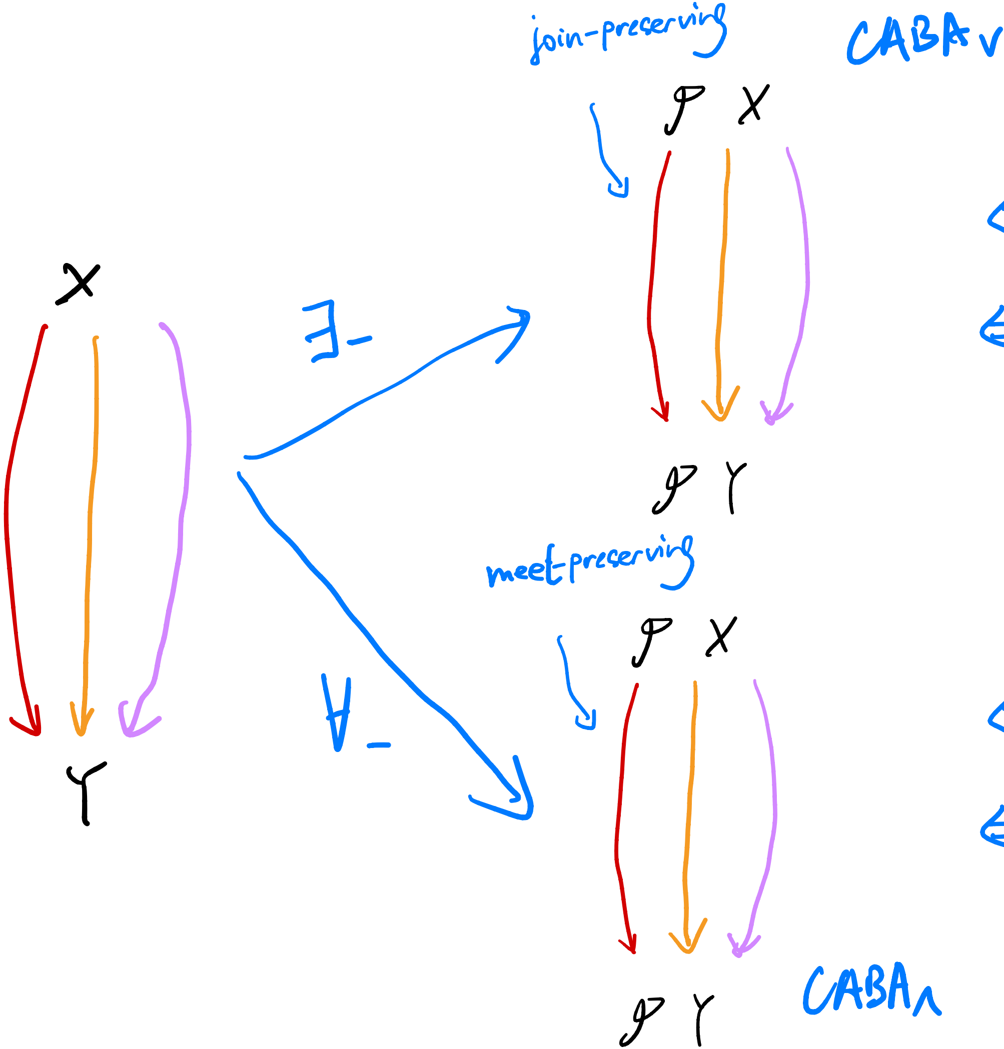
$$[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!], \qquad [\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]. \tag{2}$$

For every R, we have an adjunction:

$$\begin{tikzcd} \mathscr{P}X \arrow[r, bend left, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathscr{P}Y \arrow[l, bend left, "\forall_{R^\dagger}"] \end{tikzcd}$$

i.e. $\exists_R(S_1) \subseteq S_2$ iff $S_1 \subseteq \forall_{R^\dagger}(S_2)$

(also $\exists_{R^\dagger} \dashv \forall_R$ via $R^\dagger$)



$$R_1 \subseteq R_2 \iff \exists_{R_1} \leq \exists_{R_2} \iff \underset{\Diamond_{R_1}}{\underset{\parallel}{\exists_{R_1^+}}} \leq \underset{\Diamond_{R_2}}{\underset{\parallel}{\exists_{R_2^+}}}$$

$$R_1 \subseteq R_2 \iff \forall_{R_2} \leq \forall_{R_1} \iff \underset{\Box_{R_2}}{\underset{\parallel}{\forall_{R_2^+}}} \leq \underset{\Box_{R_1}}{\underset{\parallel}{\forall_{R_1^+}}}$$

$\Rightarrow R$

$\Diamond_R \overset{\text{dual}}{\Longleftrightarrow} R \overset{\text{dual}}{\Longleftrightarrow} \Box_R$

Examples of correspondence results

- R is refl iff $1_X \subseteq R$ iff $\Box_R \leq 1_{\mathcal{P}X}$

$$\Updownarrow$$

$$\Box\varphi \vdash \varphi.$$

- R is trans iff $R \circ R \subseteq R$ iff $\Box_R \leq \Box_R \circ \Box_R$

$$\Updownarrow$$

$$\Box\varphi \vdash \Box\Box\varphi.$$

Categories of Kripke Frames

$$\begin{array}{c}(X, R_X)\\ \downarrow f \\ (Y, R_Y)\end{array}$$

monotone map preserves relations:

$wR_Xv \Rightarrow f(w) R_Y f(v)$.

(forth)

also (+back) condition

$\boxed{Kr}$ $\boxed{Kr_B}$

$$Rel \overset{\text{dual}}{\Longleftrightarrow} CABA_s$$

$X \longmapsto \mathcal{P}(X)$

$R \longmapsto \Diamond_R$

$$\Downarrow$$

$$Kr \overset{\text{dual}}{\Longleftrightarrow} CABAO_s$$

$(X, R) \longmapsto (\mathcal{P}(X), \Diamond_R)$

Kr is topological over sets !

Fact 3. **Kr** is "topological over **Sets**",[15] meaning, concretely, the following. Given any family of functions $f_i : X \to Y_i$ $(i \in I)$ to Kripke frames (Y_i, R_i), the relation

$$wR_X v \iff f_i(w) R_i f_i(v) \text{ for all } i \in I, \quad \text{i.e.} \quad R_X = \bigcap_{i \in I} (f_i^{\dagger} \circ R_i \circ f_i),$$

is the (unique) "initial lift" of $\{f_i\}_{i \in I}$, i.e. the relation on X such that, given any function $g : Z \to X$, all $f_i \circ g$ are monotone from a frame (Z, R_Z) iff g is.

(In fact, Fact 3 holds of **Kr** in general, again with R^α in place of R_X.) One may note that the relation

$$\begin{tikzcd}
 & (X, \underline{?}) \arrow[dl, "f_1"'] \arrow[d, "f_n"] & \underline{R_X} \arrow[l, bend right] \\
(Y_1, R_1) & (Y_n, R_n) &
\end{tikzcd}$$

- -

the largest/coarsest relation preserved by all f_i.

Moreover,

- Initial lifts preserve refl, trans, symm. ...

$\Rightarrow$ Subcats like Preord, Equiv are initially closed.

$\Rightarrow$ They are also topological over Sets.

inclusion functors have left adjoints :

e.g.
$$\begin{tikzcd} \mathbf{Kr} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathbf{Preord} \arrow[l, bend right, "i"'] \end{tikzcd}$$

$$(X, R) \longmapsto (X, R^{*})$$

Also final lifts:

e.g.

$$
\begin{tikzcd}
& (X, \underline{?}) & \\
(X, R_1) \arrow[ur, "1_X"] & \cdots & (X, R_n) \arrow[ul, "1_X"']
\end{tikzcd}
$$

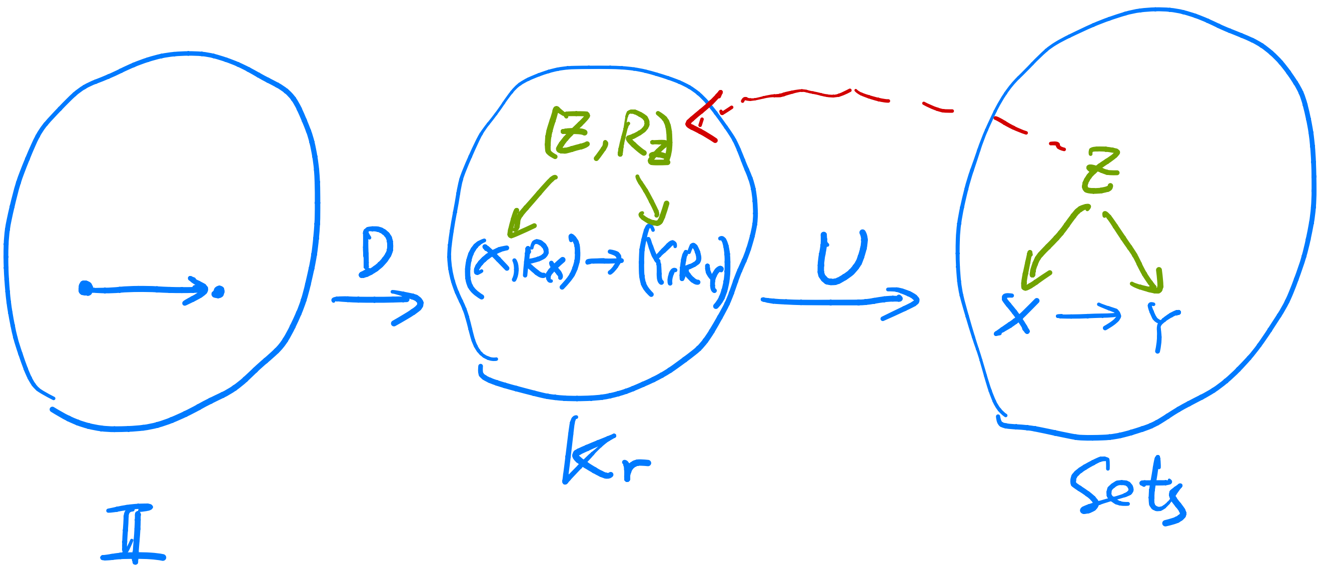
$R_X = (\bigvee_\alpha R_\alpha)^*$

In Preord

So common knowledge is no ad hoc construction!

We only need to look at (co)limits in Sets!

Another consequence, more relevant to this article, is that the forgetful functor $U : \mathbf{Kr} \to \mathbf{Sets}$ to the complete and cocomplete category **Sets** lifts limits and colimits—meaning that, given any (small) diagram D in **Kr**, its (co)limit exists on the (co)limit of $U \circ D$ in **Sets**. Most notably,



Examples.

$$\prod_{i\in I}(Y_i) = (X, \underline{?}) \qquad \underline{R_X} := \bigcap_{i\in I}\left(p_i^{+} \circ R_i \circ p_i\right)$$

the largest/coarsest relation preserved by all f_i.

$$\begin{tikzcd}
 & (X, \underline{?}) \arrow[dl, "p_1"'] \arrow[d, "p_n"] \\
(Y_1, R_1) \quad \cdots & (Y_n, R_n)
\end{tikzcd}$$

$$\underbrace{(S, R_S)}_{\text{subframe on } S} \xrightarrow{\ i\ } (X, R_X)$$

Remarks:

These canonical maps are not in general bounded morphisms. So we are not working in Kr_B.

Second Part

Semantics of DEL

PAL first:

unary operators:

$[\sigma!]$, $\langle\sigma!\rangle$

$[\sigma!]\varphi$

$\downarrow$

φ will be the case after σ is publicly and truthfully announced (observed)

P A L

A public announcement of σ

$$(X, R_X, [\![\ ]\!]_X)$$

$$S = [\![\sigma]\!]_X$$

$$(S, R_S) \xrightarrow{i} (X, R_X)$$

The reduction axioms follow!

$$[\![\, [\sigma!]\varphi \,]\!]_X = \forall_i [\![\varphi]\!]_S .$$

($w \in [\![\, [\sigma!]\varphi \,]\!]_X$ iff $v \in [\![\varphi]\!]_S$ whenever $v i w$)

$$[\![\sigma \Rightarrow \varphi]\!]_X = \forall_i \circ i^{-1} [\![\varphi]\!]_X .$$

At atoms level, we have

$$[\![p]\!]_S = i^{-1} [\![p]\!]_X$$

So we get the reduction axiom:

$$[\![\, [\sigma!] p \,]\!]_X = [\![\sigma \Rightarrow p]\!]_X$$

Another reduction axiom:

The dual of $R_s \circ i^t = i^t \circ R_X \circ i \circ i^t$

$$\Downarrow$$

$$\forall_i \circ \forall_{R_s^t} [\![\varphi]\!]_s = \forall_i \circ i^{-1} \circ \forall_{R_X^t} \circ \forall_i [\![\varphi]\!]_s$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$[\![[\sigma!] \Box \varphi]\!]_X \qquad\qquad [\![\sigma \Rightarrow \Box [\sigma!] \varphi]\!]_X$$

Remark

$[\sigma!] \sim \forall_i$ similar to $\Box \sim \forall_{R^t}$ but PAL (DEL) generalizes Kripke semantics by using $X \leftrightarrow Y$ between different Kripke models to interprete modal operators.

DEL

Event update:

Epistemic model: (X, R_X) with $[\![\mathrm{Pre}(e)]\!]_X \subseteq X$ for $\forall e \in E$.

Event model: (E, R_E)

$$X \otimes E = \sum_{e \in E} [\![\mathrm{Pre}(e)]\!]_X = \{(w, e) \in X \times E \mid w \in [\![\mathrm{Pre}(e)]\!]_X\}$$

$$
\begin{tikzcd}
X \otimes E = \sum_{e \in E} [\![\mathrm{Pre}(e)]\!]_X \arrow[r, hook, "i"] \arrow[rr, bend left, "p_E"] \arrow[dr, bend left, "p_X"] & X \times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
[\![\mathrm{Pre}(e)]\!]_X \arrow[u, "q_e"] \arrow[r, hook, "i_e"'] \arrow[ur, phantom, "\urcorner", very near start] & X \arrow[u, shift left, "q_{e'}"] \arrow[ul, "R_e", "/"{marking}] &
\end{tikzcd}
$$

coproduct injections

$R_{X \otimes E}$ is the initial lift of p_X and p_E.

Tabulation: $R_e = q_e \circ i_e^{+}$

Similar as before:

$$[\![[E,e]\varphi]\!]_X = \forall_{R_e^+} [\![\varphi]\!]_{X\otimes E}$$

$$[\![\langle E,e\rangle\varphi]\!]_X = \exists_{R_e^+} [\![\varphi]\!]_{X\otimes E}$$

and the reduction axioms follow due to some duality.

Example:

$$[\![\, [E,e]\ \Box_b(\Box_a p \vee \Box_a \neg p)\,]\!]_X$$

$$= \forall_{R_e^+}[\![\Box_b(\Box_a p \vee \Box_a \neg p)]\!]_{X\otimes E}$$

$$= \forall_{R_e^+}\left(\forall_{R^+_{b,X\otimes E}}[\![\Box_a p \vee \Box_a \neg p]\!]_{X\otimes E}\right)$$

$$= \forall_{R_e^+}\Big(\forall_{R^+_{b,X\otimes E}}\big(\overbrace{\forall_{R^+_{a,X\otimes E}}[\![p]\!]_{X\otimes E}}^{\{(w_1,e_1)\}}$$

$$\cup\ \underbrace{\forall_{R^+_{a,X\otimes E}}(X\otimes E\setminus[\![p]\!]_{X\otimes E})}_{\{(w_2,e_2)\}}\big)\Big)$$

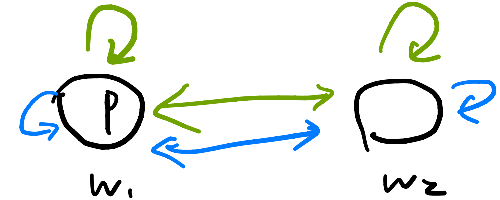
$$= \forall_{R_e^+}\left(\forall_{R^+_{b,X\otimes E}}(\{(w_1,e_1),(w_2,e_2)\})\right)$$

$R_e = \{e\circ !\}^+ : X \nrightarrow X\otimes E$

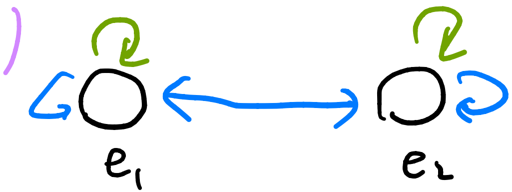
$$= \forall_{R_e^+}\{(w_1,e_1),(w_2,e_2)\}$$

$$= \{w_1,w_2\}$$

(X, R_X)

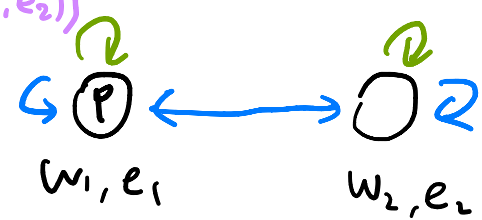


(E, R_E)



$Pre(e_1) = p$ $Pre(e_2) = \neg p$

$\Downarrow$



$\rightarrow$: a's relation

$\rightarrow$: b's relation

Some remarks:

We conclude this section with a remark on the significance of using the category **Kr**. We reviewed in this section that topological constructions (Subsection 2.4) and their canonical maps play essential rôles in the semantics of PAL and DEL. These constructions take place in **Kr** as opposed to the category $\mathbf{Kr_B}$, and the canonical maps are monotone maps of **Kr**, and not bounded morphisms of $\mathbf{Kr_B}$. Indeed, for

DEL to show interesting behaviors, the canonical maps—in particular, $p_X : X \otimes E \to X$, which amounts to $i : [\![\sigma]\!]_X \hookrightarrow X$ in the case of PAL—must not be bounded moprphisms. For, if p_X is a bounded morphism, then $[\![\varphi]\!]_{X\otimes E} = {p_X}^{-1}[\![\varphi]\!]_X$ for every φ and not just atomic p (this entails $[E, e]\varphi \equiv \mathrm{Pre}(e) \Rightarrow \varphi$ the same way as in (26))—this means that no event can teach agents anything. In other words, for events to teach agents something, they must bring about some change logically, and therefore the maps f representing them must not have logic-preserving duals f^{-1}.

Related to two previous questions.

- Why do we need monotone but not bounded morphisms for dynamic updates?

Monotone:

$$f \circ R_X \subseteq R_Y \circ f$$

$(w R_X v \Rightarrow f(w) R_Y f(v))$

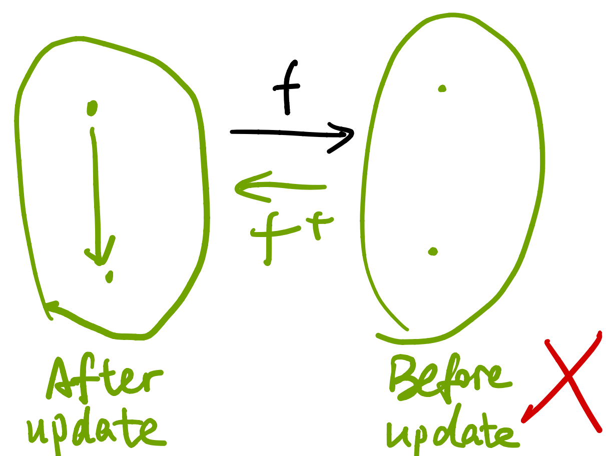
The agent cannot 'unknow'!

Bounded:

$$f \circ R_X = R_Y \circ f$$

(bisimulation)

too strict



Bounded morphisms imply that no event can teach agents anything (see the event update example later).

p_X is bounded

$\Rightarrow [\![\varphi]\!]_{X \otimes E} = p_X^{-1} [\![\varphi]\!]_X$ for every φ

$\Rightarrow [E, e]\varphi \equiv pre(e) \Rightarrow \varphi$ (including modal formulas!)

- What do we need tabulations for?

  In Kripke semantics, we use relations to interpret modal operators:

Kripke semantics uses binary relations to interpret unary modal operators. A Kripke frame is a set X paired with a binary relation $R : X \nrightarrow X$, and a Kripke model is a Kripke frame (X, R) equipped with an assignment $[\![-]\!]$ of subsets $[\![p]\!] \subseteq X$ to propositional variables p. In fact we extend the notation to all propositions φ, so that $w \in [\![\varphi]\!] \subseteq X$ means that φ is true at w. Now, given a relation $R : X \nrightarrow Y$, define two monotone maps $\exists_R, \forall_R : \mathscr{P}X \to \mathscr{P}Y$ by

$$\exists_R(S) = \{ v \in Y \mid w \in S \text{ for some } w \in X \text{ such that } wRv \}, \quad \longrightarrow \Diamond_R = \exists_{R^\dagger}$$

$$\forall_R(S) = \{ v \in Y \mid w \in S \text{ for all } w \in X \text{ such that } wRv \}. \quad \longrightarrow \Box_R = \forall_{R^\dagger}$$

for $R : X \nrightarrow X$

Then, for a relation $R : X \nrightarrow X$ on a set X, $\exists_{R^\dagger}, \forall_{R^\dagger} : \mathscr{P}X \to \mathscr{P}X$ interpret the "possibility" operator $\Diamond$ and the "necessity" operator $\Box$, respectively—i.e.

$$[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!], \qquad [\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]. \tag{2}$$

But in Kr.

$$(X, R_X) \xrightarrow{R} (Y, R_Y)$$

R is not a morphism. But we may need to use R for defining dynamic operators as across-model modalities.

$$
\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y
\end{tikzcd}
\qquad R = r_2 \circ r_1^{+}
$$

But does such (R, R_R) exist?

$$
\begin{tikzcd}
 & (R, R_R) \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
(X, R_X) & & (Y, R_Y)
\end{tikzcd}
$$

Fact 3 gives us a canonical construction of R_R.

$$R = r_2 \circ r_1^{\dagger}$$

$$\forall_{R^{\dagger}} = \forall_{r_1} \circ r_2^{-1}, \quad \exists_{R^{\dagger}} = \exists_{r_1} \circ r_2^{-1}$$

(See the event update example for a real use)

$$
\begin{tikzcd}
X \otimes E = \sum_{e\in E} [\![Pre(e)]\!]_X \arrow[rr, "i"] \arrow[rrd, "p_X"] \arrow[rrr, bend left, "p_E"] & & X\times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
[\![Pre(e)]\!]_X \arrow[u, "q_e"] \arrow[rr, "i_e"'] \arrow[urr, phantom, "\urcorner", very near start] & & X \arrow[u, shift left, "q_{e'}"] \arrow[llu, "R_e"', "|" marking] &
\end{tikzcd}
$$

coproduct injections

$R_{X\otimes E}$ is the initial lift of p_X and p_E.

$$[\![[E,e]\varphi]\!]_X = \forall_{R_e^\dagger} [\![\varphi]\!]_{X\otimes E}$$

$$[\![\langle E,e\rangle \varphi]\!]_X = \exists_{R_e^\dagger} [\![\varphi]\!]_{X\otimes E}$$

Tabulation: $R_e = q_e \circ i_e^\dagger$

Applications

Applications:

Integration with FOL:

FOL + ML

(semantics in slice category $Sets/X$) (Kripke semantics)

$\Downarrow$

FO-ML

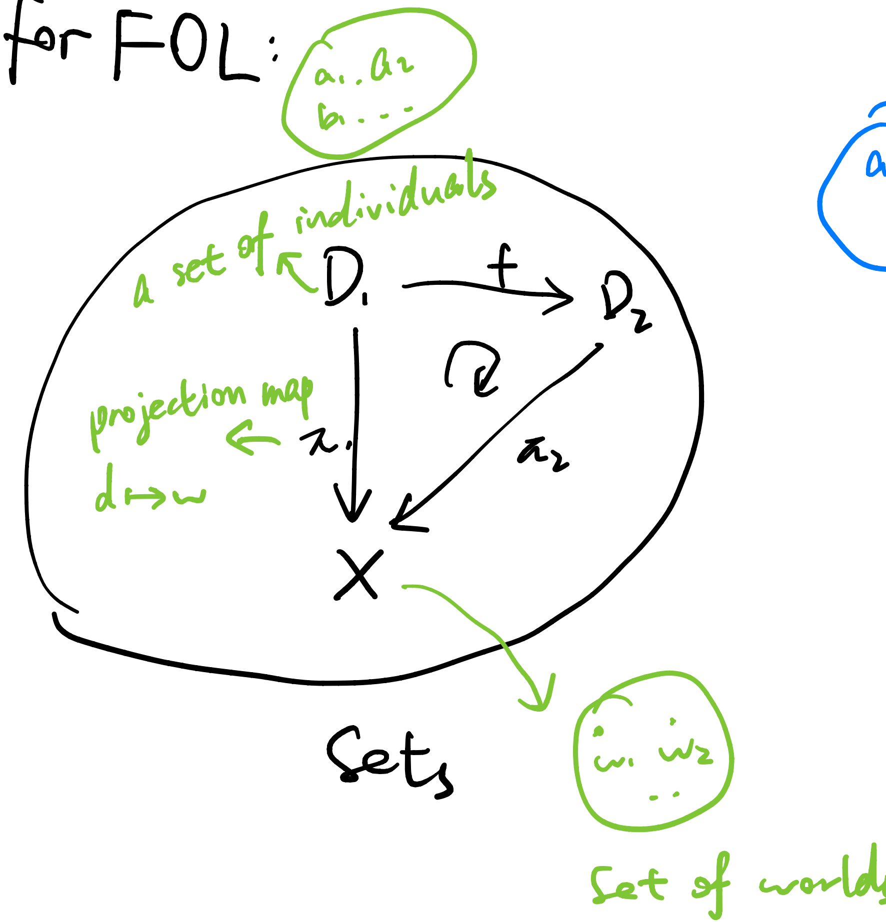
(Kripke sheaves) glues together local structures over worlds consistently

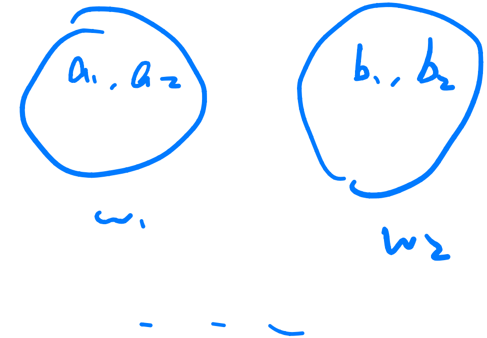
$\Downarrow$ + update

FO-DEL

(new sheaf semantics)

Semantics for FOL:





Slice category:

$\boxed{\mathbf{Sets}/X}$

$D_X^2 = \{ (a_1, a_1), (a_1, a_2), (b_1, b_1) \cdots \}$

but $\not\ni (a_1, b_1)$ as they are in different worlds.

(H,R). Its objects and arrows are monotone maps and not just any functions.

Fixing any (nonempty) set X, the slice category **Sets**$/X$ is used to interpret classical first-order logic as follows. We fix an object $\pi : D \to X$ of **Sets**$/X$, and a surjection π in particular. We then regard X as a set of worlds and D as a set of individuals. Each individual $a \in D$ is assumed to live in a unique world, viz. $\pi(a) \in X$. In this sense we may call π a "residence map". For each world $w \in X$, the fiber $D_w = \pi(\{w\})$ is the set of individuals living in w. In fact, for each $n \in \mathbb{N}$, the cartesian product $D_w^n = D_w \times \cdots \times D_w$ is the set of n-tuples of individuals living in w, and the disjoint union of D_w^n for all $w \in X$, i.e. the n-fold "fibered product" of D over X,

$$D_X^n = \sum_{w \in X} D_w^n = \{ (a_1, \ldots, a_n) \in D \times \cdots \times D \mid \pi(a_1) = \cdots = \pi(a_n) \},$$

is the set of n-tuples from the same world, with the projection

$$\pi^n : D_X^n \to X :: \bar{a} \to \pi(a_i)$$

mapping an n-tuple from the same world to that world. (As special cases, $D_X^1 = D$ and $D_X^0 = X$, with $\pi^1 = \pi : D \to X$ and $\pi^0 = 1_X : D \to D$.) Categorically speaking, this is to take the n-fold pullback of D over X in **Sets**, or the n-fold product of π in **Sets**$/X$.

Quantifiers:

for Boolean operators. For quantifiers, take a projection $p : D_X^{n+1} \to D_X^n :: (\bar{a}, b) \mapsto \bar{a}$ and let

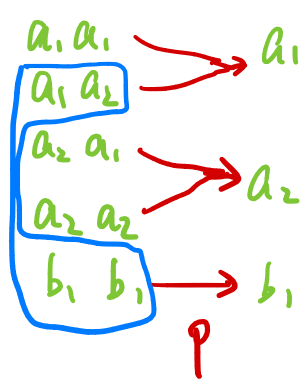
$$[\![\bar{x} \mid \forall y.\, \varphi]\!] = \forall_p [\![\bar{x}, y \mid \varphi]\!], \qquad [\![\bar{x} \mid \exists y.\, \varphi]\!] = \exists_p [\![\bar{x}, y \mid \varphi]\!]; \tag{30}$$

the case of $n = 0$ is just $p = \pi : D \to X$. Closely connected to quantification is the substitution of terms:

Example:

$\rightarrow : S(-,-)$

$$[\![x \mid \forall y.\, S(x,y)]\!]$$
$$= \forall_p [\![x, y \mid S(x,y)]\!]$$
$$= \forall_p \{(a_1, a_2), (b_1, b_1)\}$$
$$= \{b_1\}$$



Kripke-sheaf semantics for FOML

Definition 1. A bounded morphism $\pi : (D, R_D) \to (X, R_X)$ is called a *Kripke sheaf* over (X, R_X) if

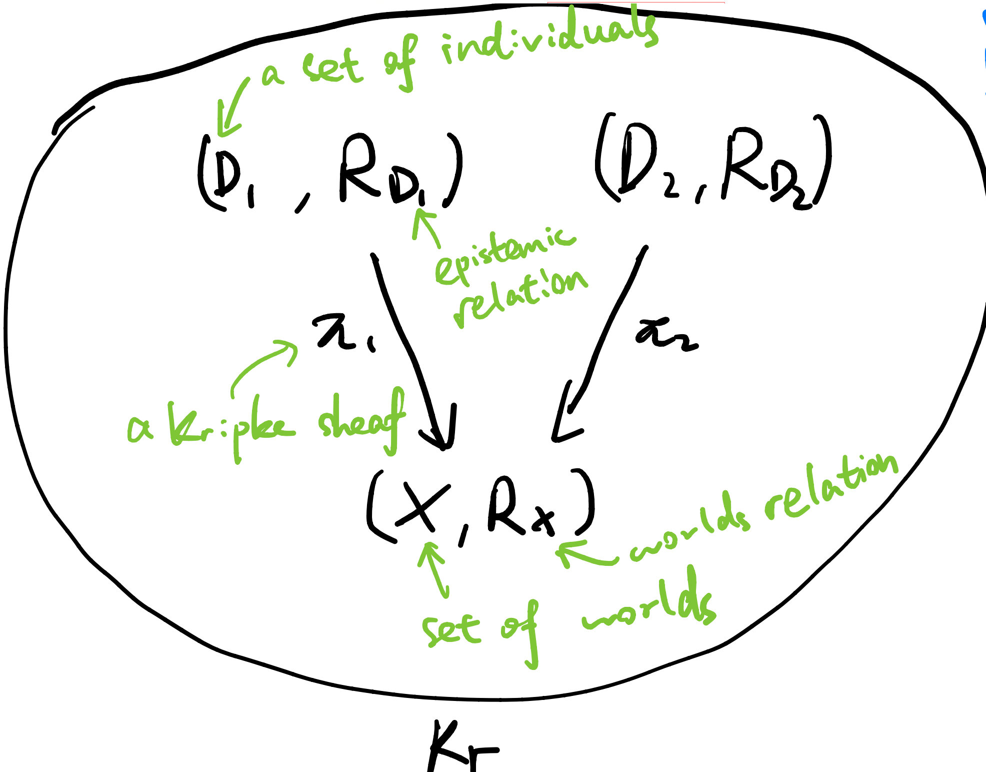
33. $aR_D b\pi w$ and $aR_D b'\pi w$ imply $b = b'$. That is, $(R_D \circ R_D{}^{\dagger}) \cap (\pi^{\dagger} \circ \pi) \subseteq 1_D$.

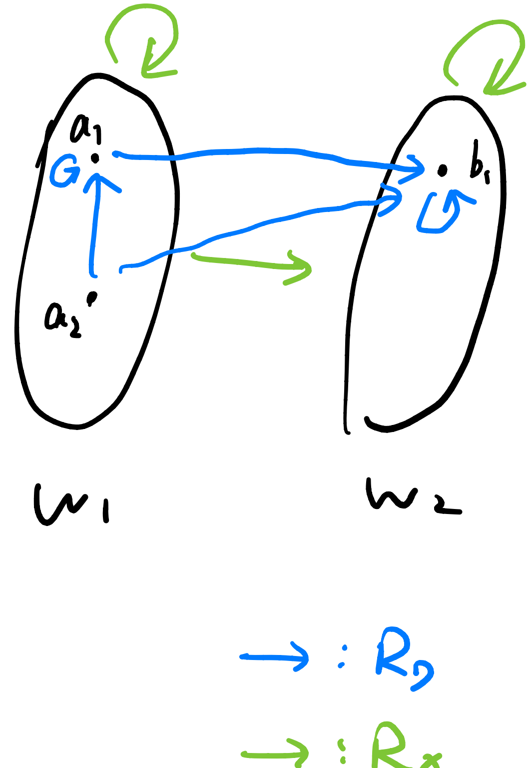
$a R_D b \Rightarrow \pi(a) R_X \pi(b)$

&

$w_1 R_X w_2 \Rightarrow \forall a \in w_1,$
$\exists b.\ a R_D b.$

$\boxed{\text{Glue}}$





Modalities:

We interpret first-order modal logic with π and other structure in $\mathbf{Kr}/(X,R_X)$. The classical base of the logic is interpreted with the underlying, non-Kripke structure in $\mathbf{Sets}/X$, just as in Subsection 4.1. The new, modal part is then added to the base using the Kripke structure, as follows: First we require that, for each n-ary function symbol f, its interpretation $[\![f]\!] : (D^n_X, R_{D^n_X}) \to (D, R_D)$ be monotone, so that all interpretations $[\![\bar{y} \mid t]\!]$ of terms are monotone—i.e., they must be arrows of $\mathbf{Kr}/(X,R_X)$. Then we set

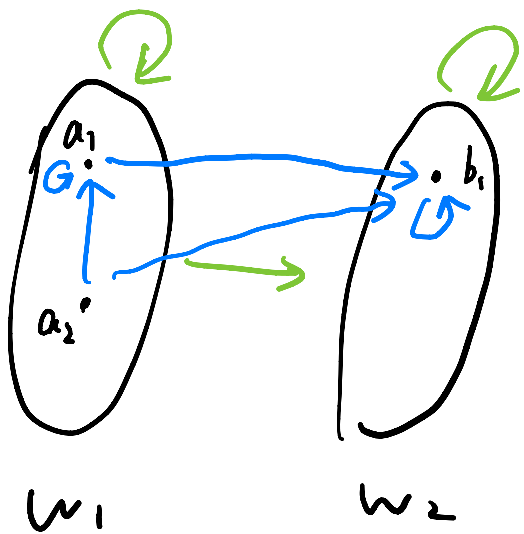
$$[\![\bar{x} \mid \Box\varphi]\!] = \forall_{R_{D^n_X}{}^\dagger}[\![\bar{x} \mid \varphi]\!], \qquad [\![\bar{x} \mid \Diamond\varphi]\!] = \exists_{R_{D^n_X}{}^\dagger}[\![\bar{x} \mid \varphi]\!]. \tag{34}$$

Each $(D^n_X, R_{D^n_X})$ is a Kripke frame.

the n-fold fibered product of D over X

the n-fold pullback of D over X

$$\begin{tikzcd}
& (D^n_X, ?) \arrow[dl] \arrow[dr] \arrow[dd, "\pi^n"] & \\
(D, R_D) \arrow[dr, "\pi"'] & & (D, R_D) \arrow[dl, "\pi"] \\
& (X, R_X) &
\end{tikzcd}$$



$(a_1, a_2) R_{D^2_X} (b_1, b_1)$

$\rightarrow : R_D$

$\rightarrow : R_X$

Example:

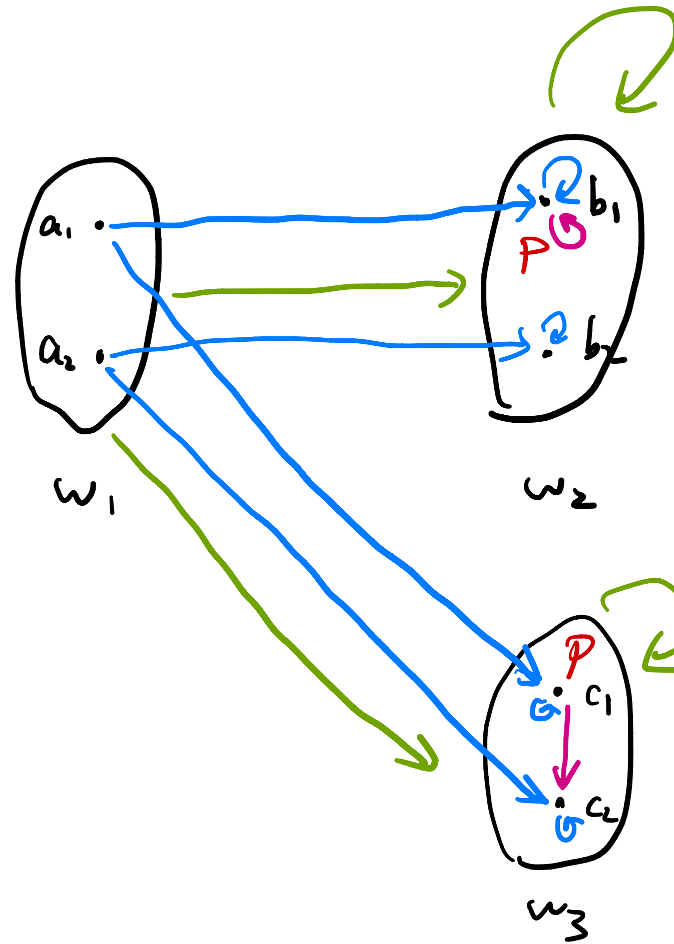
$$X = \{w_1, w_2, w_3\}$$

$$D = \{a_1, a_2, b_1, b_2, c_1, c_2\}$$

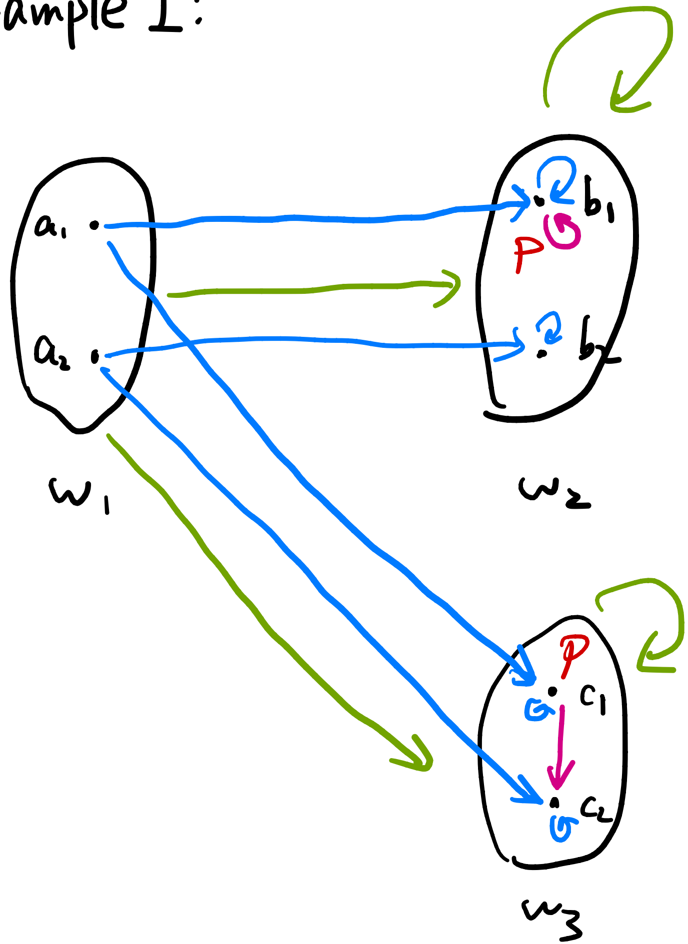
$\rightarrow$: R_X worlds relation

$\rightarrow$: R_D epistemic relation

$\rightarrow$: $S(-,-)$



Example 1:



$$[\![\Box(\exists x P(x))]\!]$$

$$= \forall_{R_{D_x^{n}}^{\dagger}} [\![\exists x P(x)]\!] \qquad (n = 0)$$

$$= \forall_{R_x^{\dagger}} [\![\exists x P(x)]\!]$$

$$= \forall_{R_x^{\dagger}} \circ \exists_{p} [\![x \mid P(x)]\!]$$

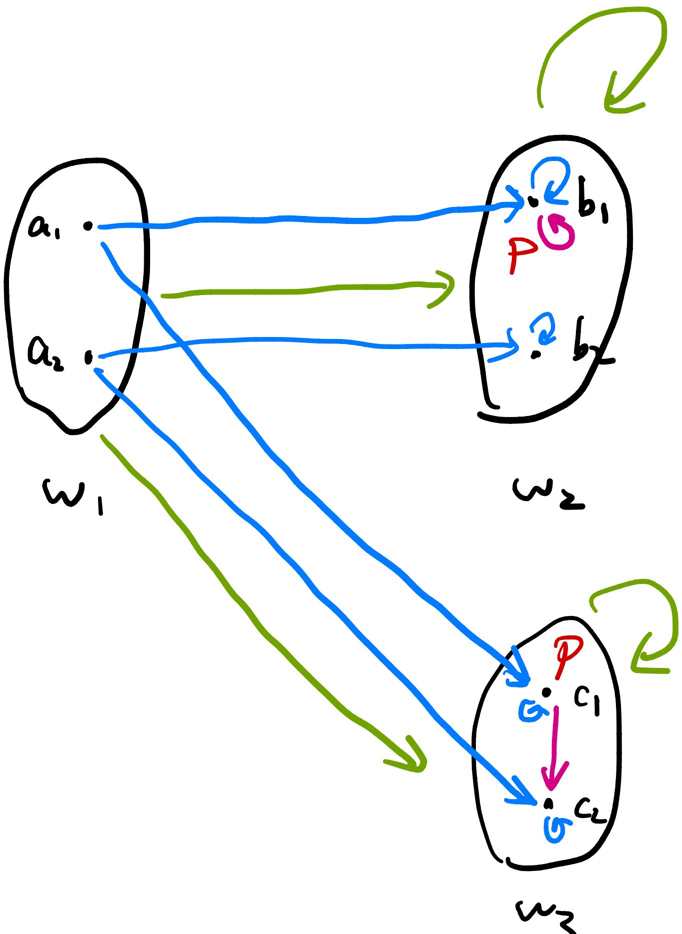
$$p = \pi : D \to X$$

$$= \forall_{R_x^{\dagger}} \circ \exists_{\pi} \{ b_1, c_1 \}$$

$$= \forall_{R_x^{\dagger}} \{ w_2, w_3 \}$$

$$= \{ w_1, w_2 \}$$

Example 2:



$(a_1, a_2)\, R_{D_x^2}\, (b_1, b_1)$

$$[\![x \mid \forall y\ \Diamond S(x,y)]\!]$$

$$= \forall_p [\![x, y \mid \Diamond S(x,y)]\!]$$

$$= \forall_p \circ \exists_{R_{D_x}^{n} +} [\![x, y \mid S(x,y)]\!] \qquad (n = 2)$$

$$= \forall_p \circ \exists_{R_{D_x^2} +} \{ (b_1, b_1), (c_1, c_2) \}$$

$(a, b) \mapsto a$

$$= \forall_p \{ (a_1, a_1), (a_1, a_2), (b_1, b_1), (c_1, c_2) \}$$

$$= \{ a_1 \}$$

$\neg, \wedge, \vee$

term-substitution

$\exists, \forall$ $\Box, \Diamond$

In short, the simple combination of (28)–(32), for classical first-order logic, and (29) and (34), for propositional modal logic, is made possible by Kripke sheaves and Fact 4. And this simple combination makes the logic of Kripke-sheaf semantics the simple union of classical first-order logic and modal logic.

Fact 5. Let **FOK** be the first-order modal logic that consists of all the rules and axioms of classical first-order logic, and the rules and axioms of propositional modal logic **K**. Then **FOK** is sound and complete with respect to the Kripke-sheaf models. The same holds with **S4** (or **S5**, respectively) in place of **K**, with respect to the subclass of Kripke-sheaf models over preorders (or equivalence relations).[23]

An advantage of categorical semantics?

FOL + ML

(semantics in slice category $\mathbf{Sets}/X$)

(Kripke semantics)

$\Downarrow$

FO-ML

(Kripke sheaves) → glues together local structures over worlds consistently

An advantage of categorical semantics?

Integration with FOL:

FOL + ML

(semantics in slice category Sets/X)

(Kripke semantics)

$\Downarrow$

FO-ML

(Kripke sheaves)

glues together local structures over worlds consistently

$\Downarrow$ + update

FO-DEL

(new sheaf semantics)

$$
\begin{tikzcd}
X \otimes E = \sum_{e \in E} [\![\mathrm{Pre}(e)]\!]_X \arrow[r, hook, "i"] \arrow[rr, bend left, "p_E"] \arrow[dr, "p_X"] & X \times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
{[\![\mathrm{Pre}(e)]\!]_X} \arrow[u, "q_e"] \arrow[r, hook, "i_e"'] \arrow[ur, phantom, "\urcorner", very near start] & X \arrow[u, shift left, "q_e'"] \arrow[ul, bend left, "R_e", "/"{marking}] &
\end{tikzcd}
$$

coproduct injections

$R_{X \otimes E}$ is the initial lift of p_X and p_E.

Tabulation: $R_e = q_e \circ i_e^{\dagger}$

Product update in DEL

$\Downarrow$

Pullback update in FO-DEL

$$p_X : X \otimes E \to X$$

gives us a pullback functor

$$p_X^* : \mathrm{Kr}/X \to \mathrm{Kr}/X \otimes E .$$

$$\begin{tikzcd}
D_X^m \otimes E \arrow[rr] \arrow[dr] \arrow[dd] & & D_X^m \arrow[dr] \arrow[dd] & \\
& D_X^n \otimes E \arrow[rr] \arrow[dl] & & D_X^n \arrow[dl] \\
X \otimes E \arrow[rr, "p_X"] & & X &
\end{tikzcd}$$

$$\text{FO-DEL} \xleftarrow{p_X^*} \text{FO-ML}$$

$$(\lambda_{X\otimes E}, [\![-]\!]_{\lambda_{X\otimes E}}) \longleftarrow (\lambda, [\![-]\!]_\lambda)$$

Another Kripke-sheaf model

A Kripke-sheaf model

$$
\begin{tikzcd}
D^n_{X\otimes E} = \sum_{e\in E} [\![\bar{x} \mid \mathrm{Pre}(e)]\!]_X \arrow[r, hook, "i^n"] \arrow[rr, bend left, "p^{n'}_E"] \arrow[dr, "p_{D^n_X}"] & D^n_X \times E \arrow[r, "p^{n'}_E"] \arrow[d, shift left, "p_{D^n_X}'"] & E \\
[\![\bar{x} \mid \mathrm{Pre}(e)]\!]_X \arrow[u, "q^n_e"] \arrow[r, hook, "i^n_e"'] \arrow[ur, phantom, "\urcorner", very near start] & D^n_X \arrow[u, shift left, "q^n_{e'}"] \arrow[ul, "R^n_e" description, "\shortmid" marking]
\end{tikzcd}
$$

coproduct injections

Tabulation: $R^n_e = q^n_e \circ i^{n\dagger}_e$

Event modalities:

Now we have two Kripke-sheaf models, $(\pi, [\![-]\!]_\pi)$ before update and $(\pi_{X\otimes E}, [\![-]\!]_{\pi_{X\otimes E}})$ after, and we can use relations between them to interpret the DEL operators $[E,e]$ and $\langle E,e\rangle$. Here is a key idea: As in (35)–(36), each sheaf model has a Kripke model for n-ary properties, D^n_X and $D^n_{X\otimes E}$; so we treat D^n_X and $D^n_{X\otimes E}$ as the product-update structure of Subsection 3.2 that interprets the application of $[E,e]$ and $\langle E,e\rangle$ to n-ary formulas-in-contexts. Since $[\![\bar{x} \mid \mathrm{Pre}(e)]\!]_\pi = (\pi^n)^{-1}[\![\mathrm{Pre}(e)]\!]$, observe

$$D^n_{X\otimes E} = \sum_{e\in E} [\![\bar{x} \mid \mathrm{Pre}(e)]\!]_\pi = \{(\bar{a},e) \in D^n_X \times E \mid \bar{a} \in [\![\bar{x} \mid \mathrm{Pre}(e)]\!]_\pi\}$$

and note the similarity to (22). We moreover have canonical maps as with (22), viz. the projection $p_{D^n_X} : D^n_{X\otimes E} \to D^n_X$ above and, for each $e \in E$,

- The inclusion map $i^n_e : [\![\bar{x} \mid \mathrm{Pre}(e)]\!]_\pi \hookrightarrow D^n_X$.
- The coproduct injection $q^n_e : [\![\bar{x} \mid \mathrm{Pre}(e)]\!]_\pi \to D^n_{X\otimes E} :: \bar{a} \mapsto (\bar{a},e)$.

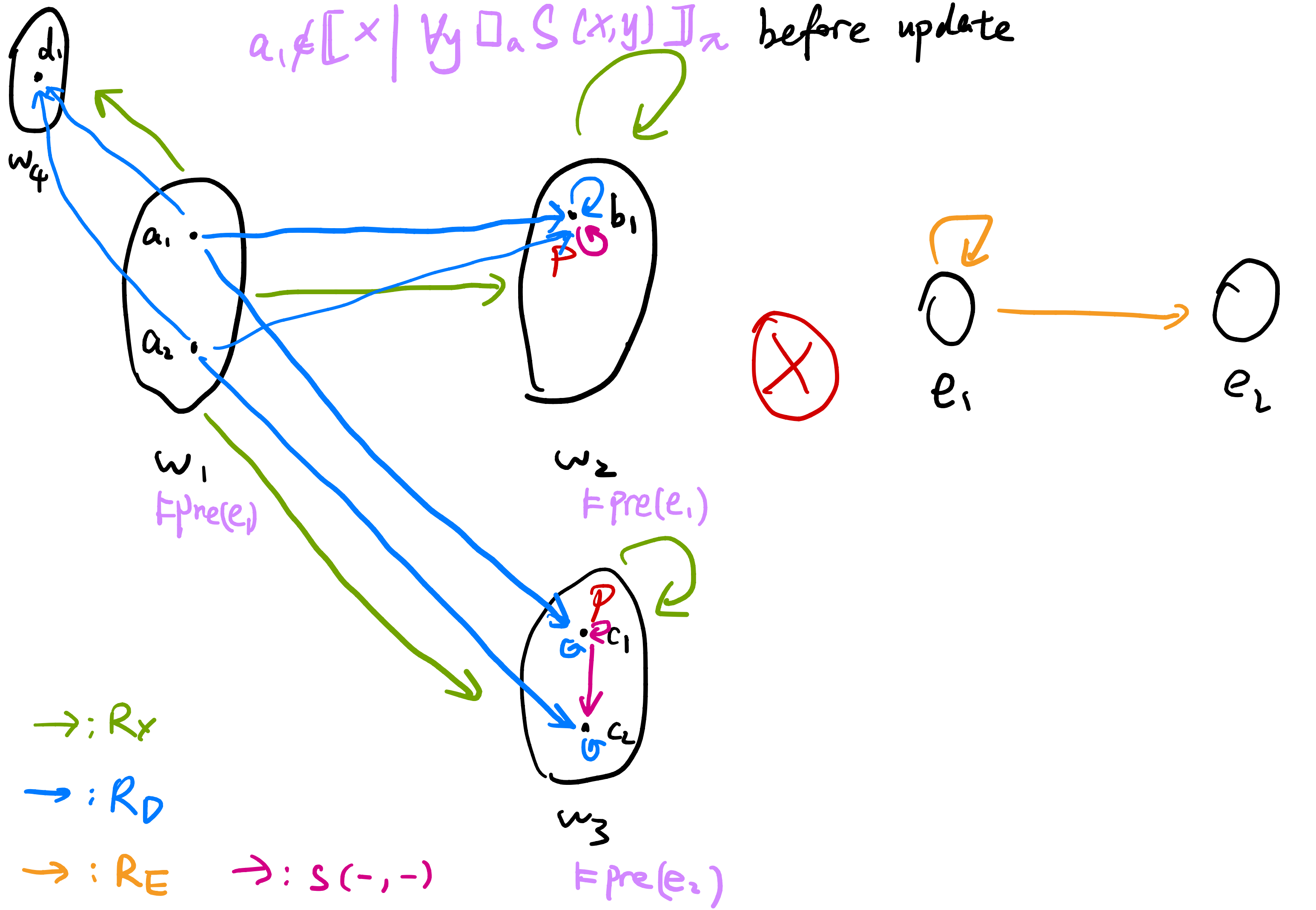
These maps tabulate a relation, $R^n_e = q^n_e \circ i^{n\dagger}_e : D^n_X \nrightarrow D^n_{X\otimes E}$, which is dual to the two maps

$$\forall_{R^{n\dagger}_e} = \forall_{i^n_e} \circ (q^n_e)^{-1}, \exists_{R^{n\dagger}_e} = \exists_{i^n_e} \circ (q^n_e)^{-1} : \mathscr{P}(D^n_{X\otimes E}) \to \mathscr{P}(D^n_X).$$

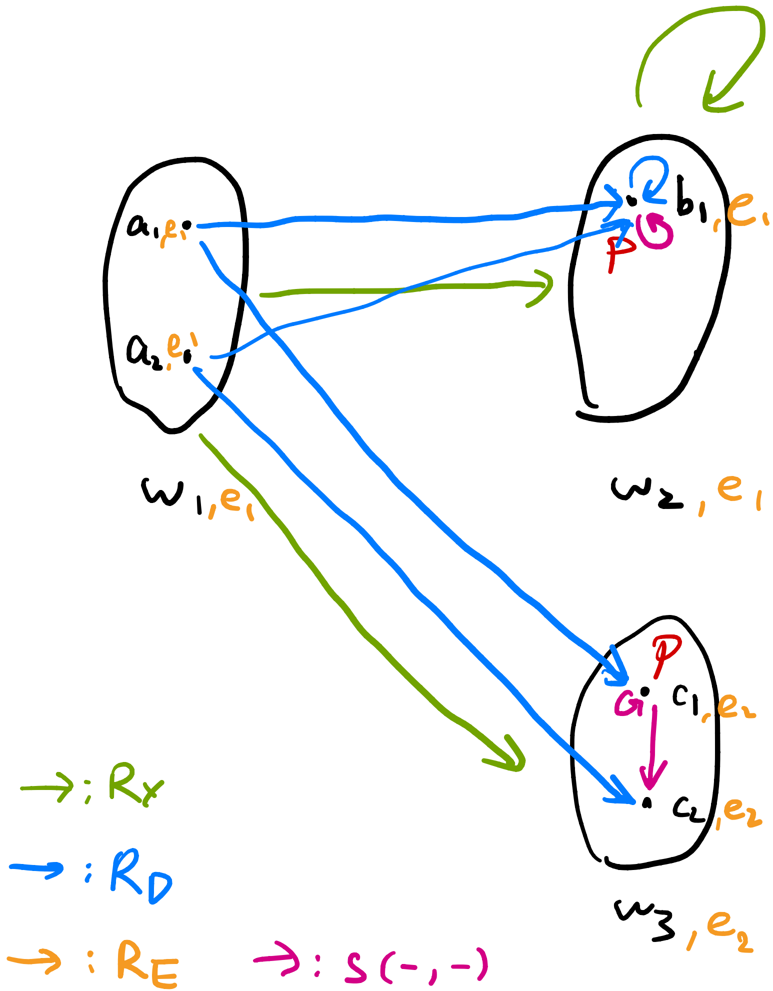
These then interpret $[E,e]$ and $\langle E,e\rangle$ applied to n-ary formulas-in-contexts $(\bar{x} \mid \varphi)$, i.e.,

$$[\![\bar{x} \mid [E,e]\varphi]\!]_\pi = \forall_{R^{n\dagger}_e}[\![\bar{x} \mid \varphi]\!]_{\pi_{X\otimes E}}, \qquad [\![\bar{x} \mid \langle E,e\rangle\varphi]\!]_\pi = \exists_{R^{n\dagger}_e}[\![\bar{x} \mid \varphi]\!]_{\pi_{X\otimes E}},$$

which is just an "in context" version of (24).



$a_1 \notin [\![x \mid \forall y \Box_a S(x,y)]\!]_\pi$ before update
d_1
w_4
a_1
a_2
w_1
$\vDash pre(e_1)$
b_1
p
w_2
$\vDash pre(e_1)$
p
c_1
c_2
w_3
$\vDash pre(e_2)$
e_1
e_2
: R_X
: R_D
: R_E
: $S(-,-)$



$$[\![x \mid [E, e_1]\, \forall y\, \Box_a S(x,y)]\!]_{\lambda}$$

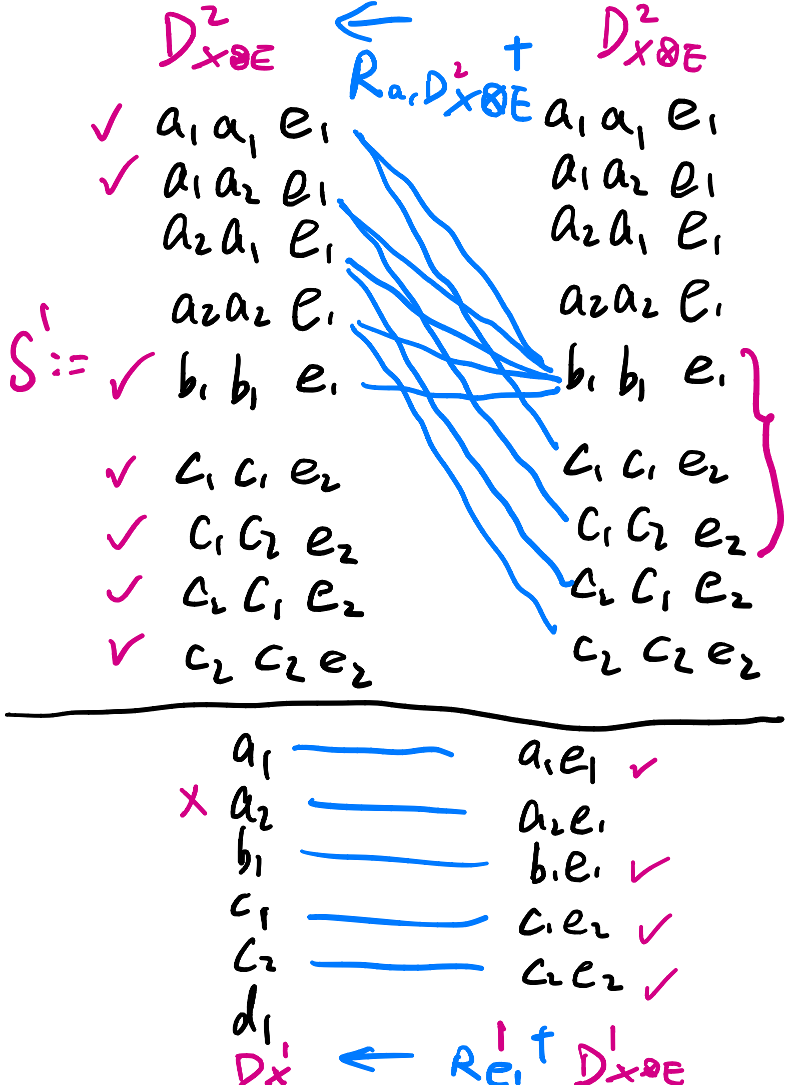
$$= \forall R_{e_1}^{n+} [\![x \mid \forall y\, \Box_a S(x,y)]\!]_{\lambda X\otimes E}$$

$$= \forall R_{e_1}^{n+} \circ \forall p\, [\![x, y \mid \Box_a S(x,y)]\!]_{\lambda X\otimes E}$$

$$= \forall R_{e_1}^{n+} \circ \forall p \circ \forall R_{a, D_{X\otimes E}^{n}}^{+} [\![x, y \mid S(x,y)]\!]_{\lambda X\otimes E}$$

$$= \forall R_{e_1}^{n+} \circ \forall p \circ \forall R_{a, D_{X\otimes E}^{\overset{2}{\shortparallel}\, n}}^{+} \left(\{(b_1, b_1, e_1), (c_1, c_1, e_2), (c_1, c_2, e_2)\}\right)$$

$$= \cdots$$



$$[\![x \mid [E, e_1]\, \forall y\, \Box_a S(x,y)]\!]_x$$

$$= \cdots$$

$$= \forall_{R_{e_1}^{n\,t}} \circ \forall_p \circ \forall_{R_{a, D^{n \,(=2)}_{x\otimes E}}^{t}} \left(\{ (b_1,b_1,e_1), (c_1,c_1,e_2), (c_1,c_2,e_2) \} \right)$$

$$= \forall_{R_{e_1}^{n\,t}} \circ \forall_p (S')$$

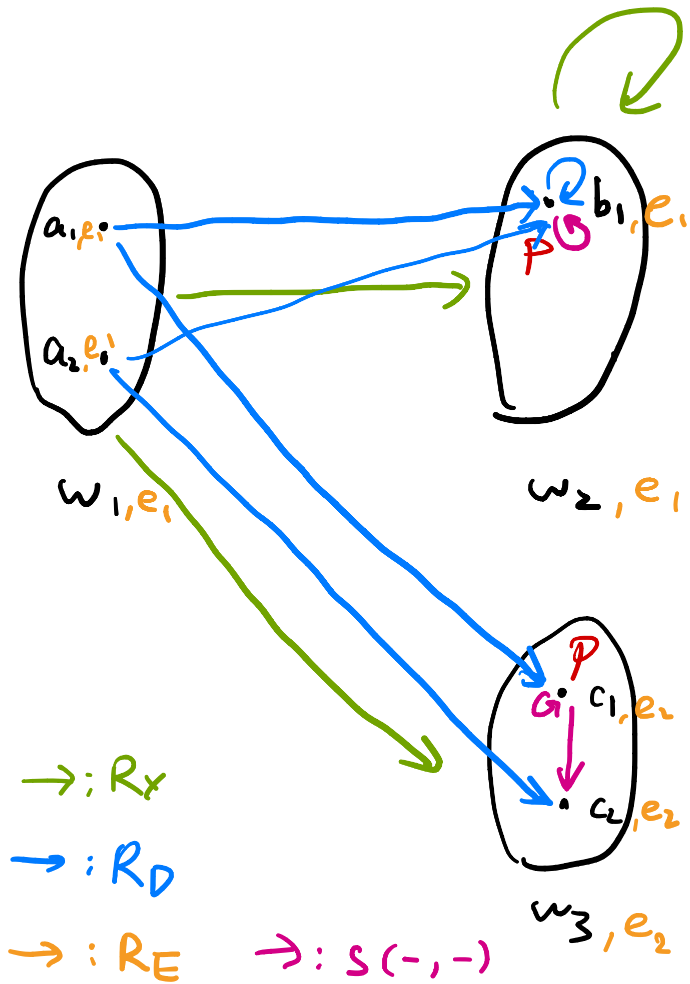
$$= \forall_{R_{e_1}^{n\,(=1)\,t}} \left(\{ (a_1,e_1), (b_1,e_1), (c_1,e_2), (c_2,e_2) \} \right)$$

$$R_e^n = q_e^n \circ i_e^{n\,t} : D_x^n \nrightarrow D^n_{x\otimes E}$$

$$= \{ a_1, b_1, c_1, c_2, d_1 \}$$

$$a_1 \in [\![x \mid [E, e_1]\, \forall y\, \Box_a S(x,y)]\!]_x$$

after update



$$[\![x \mid [E,e_1]\, \forall y\, \Box_a S(x,y)]\!]_x$$

$$= \cdots$$

$$= \forall_{R_{e_1}^{n\dagger}} \circ \forall_p \circ \forall_{R_{a,D_{x\otimes z}^{n}}^{\dagger}} \left(\{(b_1,b_1,e_1),\ (c_1,c_1,e_2),\ (c_1,c_2,e_2)\}\right)$$

$$= \forall_{R_{e_1}^{n\dagger}} \circ \forall_p (S')$$

$$= \forall_{R_{e_1}^{n\dagger}} \left(\{(a_1,e_1),\ (b_1,e_1),\ (c_1,e_2),\ (c_2,e_2)\}\right)$$

$$R_e^n = q_e^n \circ i_e^{n\dagger} : D_x^n \nrightarrow D_{x\otimes E}^n$$

$$= \{a_1, b_1, c_1, c_2, d_1\}$$

$$a_1 \in [\![x \mid [E,e_1]\, \forall y\, \Box_a S(x,y)]\!]_x$$

after update

The usual reduction axioms and

$$[E, e]\, \forall y. \varphi \equiv \forall y. [E, e]\, \varphi$$

are validate.

Theorem 4. *Let* **FODEL-K** *be the first-order modal logic that consists of* **FOK**, *all the reduction axioms of propositional DEL, and* (39). *Then* **FODEL-K** *is sound and complete with respect to the Kripke-sheaf models with pullback updates. The versions with* **S4** *and* **S5** *in place of* **K** *hold with respect to the obvious subclasses of Kripke-sheaf models.*

An advantage of categorical semantics?

Integration with FOL:

FOL + ML

(semantics in slice category Sets/X) (Kripke semantics)

$\Downarrow$

FO-ML

(Kripke sheaves) glues together local structures over worlds consistently

$\Downarrow$ + update

FO-DEL

(new sheaf semantics)

The role of $\forall_{\_}$ (and $\exists_{\_}$) in different logics:

$$[\![\Box\varphi]\!] = \forall_{R^+}[\![\varphi]\!]$$

$$[\![[\sigma!]\varphi]\!]_x = \forall_i [\![\varphi]\!]_s$$

$$[\![[E,e]\varphi]\!]_x = \forall_{R_e^+}[\![\varphi]\!]_{x\otimes E}$$

$$[\![\bar{x} \mid \forall y.\varphi]\!] = \forall_p [\![\bar{x}, y \mid \varphi]\!]$$

$$[\![\bar{x} \mid \Box\varphi]\!] = \forall_{R_{D_x^n}^{+}}[\![\bar{x} \mid \varphi]\!]$$

$$[\![\bar{x} \mid [E,e]\varphi]\!]_\pi = \forall_{R_e^{n+}}[\![\bar{x} \mid \varphi]\!]_{\pi_{X\otimes E}}$$

Connections

of these approaches and ours.

Semantics of modal logic **S4** shows various categorical structures. A Kripke frame $(X, \precsim)$ for **S4** is a preorder, and hence itself a category. Also, the family $\mathscr{O}X$ of $\precsim$-upward closed subsets of X forms a topology on X, and hence a category. Moreover, the interior operation $\mathrm{int} : \mathscr{P}X \to \mathscr{O}X$ of this topology is right adjoint to the inclusion $i : \mathscr{O}X \hookrightarrow \mathscr{P}X$, so that $\Box = i \circ \mathrm{int}$ is the comonad of the adjunction.[26] The notion of (Kripke) sheaf lifts all these structures to the first order: A Kripke sheaf over a preorder $(X, \precsim)$ is equivalently a "presheaf" on the category $(X, \precsim)$, an "étale space" over the space $(X, \mathscr{O}X)$, and a "sheaf" on the category $\mathscr{O}X$.[27] Moreover, the adjunction $i \dashv \mathrm{int}$ is lifted to a "geometric morphism" $i^* \dashv i_*$ from $\mathbf{Sets}/X$ to the "topos" of sheaves over $\mathscr{O}X$, so that its comonad $i^* \circ i_*$ induces $\Box : \mathscr{P}D \to \mathscr{P}D$ for every Kripke sheaf $\pi : (D, \precsim_D) \to (X, \precsim)$.[28] Not all these categorical structures carry over to the general (i.e. non-**S4**) Kripke semantics. It will be interesting, however, to investigate how to integrate them with DEL updates, given that epistemic relations are normally assumed to be preorders. In fact, given a monotone map $f : (X, \precsim_X) \to (Y, \precsim_Y)$ of preorders, the pullback functor f^* (which plays a key rôle in the pullback update of Subsection 4.3) has a right adjoint f_*, and $f^* \dashv f_*$ is a typical example of geometric morphism, from the topos of Kripke sheaves over $(X, \precsim_X)$ to those over $(Y, \precsim_Y)$.

A categorical approach that covers the entire Kripke semantics (for static modal logic) is given by coalgebras (see, e.g., [38, 13, 30, 23]). The category **Rel** of relations is the "Kleisli category" of the "powerset monad" $\mathscr{P} : \mathbf{Sets} \to \mathbf{Sets}$, meaning, among other things, that the relations $R : X \nrightarrow Y$ correspond 1–1 to the functions $r : X \to \mathscr{P}Y$.[29] Indeed, the powerset monad is precisely the duality $\exists_{\_} : \mathbf{Rel} \to \mathbf{CABA}_{\vee}$ restricted to **Sets** (and followed by the forgetful $U : \mathbf{CABA}_{\vee} \to \mathbf{Sets}$). The correspondence implies that the Kripke frames $(X, R : X \nrightarrow X)$ are exactly the coalgebras $r : X \to \mathscr{P}X$ for the endofunctor $\mathscr{P}$. Their homomorphisms, from $r_X : X \to \mathscr{P}X$ to $r_Y : Y \to \mathscr{P}Y$, are normally defined as functions $f : X \to Y$ satisfying $\exists_f \circ r_X = r_Y \circ f$, which amounts to (10), $f \circ R_X = R_Y \circ f$, for the corresponding relations R_X and R_Y. Therefore, in the coalgebraic approach to Kripke semantics, $\mathbf{Coalg}(\mathscr{P})$, the category of coalgebras and their homomorphisms normally considered, is—like the category **CABAO** of CABAOs and their homomorphisms—equivalent to the category $\mathbf{Kr_B}$ of bounded morphisms. In this article, on the other hand, we emphasized the significance of the topological category **Kr** of monotone maps for DEL.[30]

There have in fact been algebraic [31] and coalgebraic [5, 14] approaches to DEL. In particular, the

There have in fact been algebraic [31] and coalgebraic [5, 14] approaches to DEL. In particular, the algebraic approach by Kurz and Palmigiano [31] uses ideas closely related to those in Section 3 of this article: They observe that the product update $X \otimes E$ is a subframe of the coproduct $X \times E = \sum_{e \in E} X$, and study the dual structure, i.e. a quotient of the product $\prod_{e \in E} \mathscr{P}(X)$.[31] Kurz and Palmigiano are well aware that these constructions do not take place in $\mathbf{Kr_B}$ or **CABAO** but rather in **Kr** and $\mathbf{CABAO_C}$. They stop short, however, of studying **Kr** or $\mathbf{CABAO_C}$, saying that "for these dual characterizations to be defined, an *a priori* specification of the fully fledged category-theoretic environment in which these constructions are taken is actually not needed" ([31], 2). We, in contrast, work under the philosophy that, when one finds a good heuritstics that leads to a new result, they should study the heuritstics itself and shape it into a theory that yields more results systematically. The point of Section 4 was to demonstrate how to put to use more structures in **Kr**. It should also be stressed that we use one more category, viz. **Rel**, and take essential advantage of the fundamental relation-modality dualities of Subsection 2.2, and not just the derivative dualities of Subsection 2.3 between Kripke frames and CABAOs.

Future Work

Those mentioned by the author:

what we need (or want) semantically.

Our new application of the categorical methodology promises to be helpful on multiple fronts of the study of DEL. Naturally expected future work is to extend our approach to more vocabulary (e.g. common knowledge or μ-calculus), more types of logic (e.g. higher-order DEL or typed DEL), more structures (e.g. probability), and more general settings (e.g. intuitionistic or constructive modal logic). Various updates can be expressed as functors between categories of models, and these expressions are expected to help characterize properties of updates such as the preservation of constructions or the admitting of reduction axioms. As mentioned in Section 5, the case of **S4** can be formulated in terms of toposes. Or our structural, topological ideas on the category **Kr** of monotone maps for DEL can be used to augment the coalgebraic generalization of the subcategory $\mathbf{Kr_B}$ of bounded morphisms. One may also find, e.g., (39) too strong for their purpose, and hence need to replace the pullback update with a more flexible idea. Furthermore, although we formulated a categorical semantics, we did not mention a crucial aspect of categorical logic—viz. an interpretation $[\![-]\!]$ as a homomorphism. To cover this aspect we need to define a "syntactic category" for DEL; this will then lead to a new theory of duality.

p_X^* as a map of "toposes" of Kripke sheaves

$$[E,e]\,\forall y.\varphi \equiv \forall y.[E,e]\,\varphi$$

And more ...

justify some notions from a categorical perspective

- Two way search:
  1. explore new notions in DEL by looking at interesting categorical constructions
  2. see what categorical constructions do existing DEL notions correspond to

Kripke frame (X,R) to (X,R^*), where R^* is the reflexive and transitive closure of R.

One consequence of **Kr**, or a subcategory such as **Preord**, being topological over **Sets** is that it also has "final lifts", dual to initial lifts of Fact 3. E.g., given a family of preorders (X,R_α) $(\alpha \in A)$ on the same set X, such as "epistemic" relations R_α of agents $\alpha \in A$, consider an A-indexed family of identity maps $\{1_X\}_{\alpha \in A}$ in **Sets**; then its final lift in **Preord** comes with the epistemic relation for the "common knowledge" of the group A, i.e. $(\bigcup_\alpha R_\alpha)^*$.[17]

Another consequence, more relevant to this article, is that the forgetful functor $U : \mathbf{Kr} \to \mathbf{Sets}$ to

what about initial lift?

$(X, ?)$

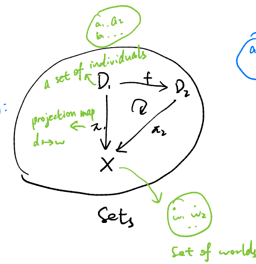
(X, R_a) (X, R_b) (X, R_c)

- Action Emulation (Malvin)
- DEL Allegories & tabulations (Benno)

- Formalization of categorical semantics in Lean?
  - Giving a unified semantics for SMCDEL?

Thanks!

Slice category: $\boxed{Sets/X}$



$$\begin{tikzcd}
& (X, ?) \arrow[dl, "f_1"'] \arrow[d, "f_n"] & R_X \arrow[l, bend right] \\
(Y_1, R_1) & \cdots \ (Y_n, R_n) &
\end{tikzcd}$$

the largest/coarsest relation preserved by all f_i.

Integration with FOL:

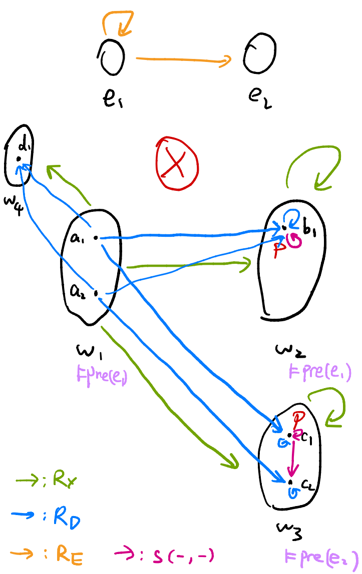
FOL (semantics in slice category $Sets/X$) + ML (Kripke semantics)

$\Downarrow$

FO-ML (Kripke sheaves) → glues together local structures over worlds consistently

$\Downarrow$ + update

FO-DEL (new sheaf semantics)



$$\begin{tikzcd}
X\otimes E=\sum_{e\in E}[\![pre(e)]\!]_X \arrow[r, hook, "i"] \arrow[dr, "p_X"] \arrow[rr, bend left, "p_E"] & X\times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
{[\![pre(e)]\!]_X} \arrow[u, "q_e"] \arrow[r, "i_e"'] & X \arrow[u, shift left, "q_{e'}"] &
\end{tikzcd}$$

coproduct injections

R_e

$[\![\Box\varphi]\!] = \forall_{R^+}[\![\varphi]\!]$

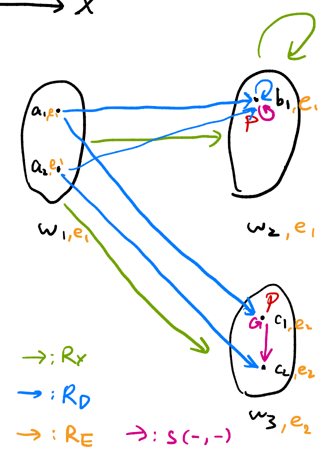
$[\![[\sigma!]\varphi]\!]_X = \forall_i[\![\varphi]\!]_s$

$[\![[E,e]\varphi]\!]_X = \forall_{R_e^+}[\![\varphi]\!]_{X\otimes E}$

$[\![\bar{x} \mid \forall y.\varphi]\!] = \forall_p[\![\bar{x},y \mid \varphi]\!]$

$[\![\bar{x} \mid \Box\varphi]\!] = \forall_{R_{D_X^n}^+}[\![\bar{x} \mid \varphi]\!]$

$[\![\bar{x} \mid [E,e]\varphi]\!]_\pi = \forall_{R_e^{n+}}[\![\bar{x} \mid \varphi]\!]_{\pi_{X\otimes E}}$



$$\begin{tikzcd}
D_X^m\otimes E \arrow[r] \arrow[dr] \arrow[dd] & D_X^m \arrow[dr] \arrow[dd] & \\
& D_X^n\otimes E \arrow[r] \arrow[dl] & D_X^n \arrow[dl] \\
X\otimes E \arrow[r, "p_X"] & X &
\end{tikzcd}$$

FO-DEL $\xleftarrow{p_X^*}$ FO-ML

$(\pi_{X\otimes E}, [\![-]\!]_{\pi_{X\otimes E}}) \longleftarrow (\pi, [\![-]\!]_\pi)$

Another Kripke-sheaf model — A Kripke-sheaf model