Categories for DEL

Goal:

- Recast the semantics of DEL in categorical terms.

  public announcements ~ pullbacks

  product updates ~ products

Benefits:

- Reveal the structural unity behind

  (the notion of common knowledge, product update, etc. is not an arbitrary or ad hoc construction since they are categorically canonical in some sense)

- Smooth integration with other logic

  (e.g. FO-DEL)

- Categorical Look at ML

- Categorical look at DEL

- Application: combine FOL and DEL

A categorical look at Kripke semantics

The category of relations

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[rr, bend left, "R", "|" marking] & & Y \arrow[loop right, "1_Y"] \arrow[ll, bend left, "R^{\dagger}", "|" marking] \end{tikzcd}$$

$\boxed{\mathbf{Rel}}$

Objects:
Sets

Morphisms:
Relations
$R \subseteq X \times Y$

Composition:

$$X \overset{R_1}{\nrightarrow} Y \overset{R_2}{\nrightarrow} Z$$

$$w \quad R_2 \circ R_1 \quad u$$

iff $\exists\, v \in Y$ s.t. $w R_1 v R_2 u$.

Examples:

- reflexivity of $R : X \nrightarrow X$

$$1_X \subseteq R$$

- transitivity of $R : X \nrightarrow X$

$$R \circ R \subseteq R$$

Facts:

- Rel is a dagger cat
- Rel is a higher cat
- Sets is a subcat of Rel
- $R \subseteq X \times Y$ can be tabulated in Sets

$$\begin{tikzcd}
& R \subseteq X \times Y \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y
\end{tikzcd}$$

s.t. $R = r_2 \circ r_1^{\dagger}$

Relation–Modality duality

Kripke semantics uses binary relations to interpret unary modal operators. A Kripke frame is a set X paired with a binary relation $R : X \nrightarrow X$, and a Kripke model is a Kripke frame (X, R) equipped with an assignment $[\![-]\!]$ of subsets $[\![p]\!] \subseteq X$ to propositional variables p. In fact we extend the notation to all propositions φ, so that $w \in [\![\varphi]\!] \subseteq X$ means that φ is true at w. Now, given a relation $R : X \nrightarrow Y$, define two monotone maps $\exists_R, \forall_R : \mathscr{P}X \to \mathscr{P}Y$ by

$$\exists_R(S) = \{v \in Y \mid w \in S \text{ for some } w \in X \text{ such that } wRv\},$$

$$\forall_R(S) = \{v \in Y \mid w \in S \text{ for all } w \in X \text{ such that } wRv\}.$$

$\Diamond_R = \exists_{R^\dagger}$

$\Box_R = \forall_{R^\dagger}$

for $R : X \nrightarrow X$

Then, for a relation $R : X \nrightarrow X$ on a set X, $\exists_{R^\dagger}, \forall_{R^\dagger} : \mathscr{P}X \to \mathscr{P}X$ interpret the "possibility" operator $\Diamond$ and the "necessity" operator $\Box$, respectively—i.e.

$$[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!], \qquad [\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]. \tag{2}$$

For every R, we have an adjunction:

$$\begin{tikzcd} \mathscr{P}X \arrow[r, bend left, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathscr{P}Y \arrow[l, bend left, "\forall_{R^\dagger}"] \end{tikzcd}$$

i.e. $\exists_R(S_1) \subseteq S_2$ iff $S_1 \subseteq \forall_{R^\dagger}(S_2)$

(also $\exists_{R^\dagger} \dashv \forall_R$ via $R^\dagger$)

$$\begin{tikzcd} X \arrow[d, bend right] \arrow[d] \arrow[d, bend left] \\ Y \end{tikzcd}$$

$\exists_{-}$ $\longrightarrow$

join-preserving

$$\begin{tikzcd} \mathcal{P}X \arrow[d, bend right] \arrow[d] \arrow[d, bend left] \\ \mathcal{P}Y \end{tikzcd}$$

$\mathbf{CABA}_{\vee}$

$\forall_{-}$ $\longrightarrow$

meet-preserving

$$\begin{tikzcd} \mathcal{P}X \arrow[d, bend right] \arrow[d] \arrow[d, bend left] \\ \mathcal{P}Y \end{tikzcd}$$

$\mathbf{CABA}_{\wedge}$

$$R_1 \subseteq R_2 \iff \exists R_1 \leq \exists R_2 \iff \underbrace{\exists R_1^{\dagger}}_{= \Diamond R_1} \leq \underbrace{\exists R_2^{\dagger}}_{= \Diamond R_2}$$

$$R_1 \subseteq R_2 \iff \forall R_2 \leq \forall R_1 \iff \underbrace{\forall R_2^{\dagger}}_{= \Box R_2} \leq \underbrace{\forall R_1^{\dagger}}_{= \Box R_1}$$

$\Longrightarrow R$

$\Diamond R \overset{\text{dual}}{\Updownarrow} R \overset{\text{dual}}{\Updownarrow} \Box R$

Examples of correspondence results

- R is refl iff $1_X \subseteq R$ iff $\Box_R \leq 1_{\mathcal{P}X}$

$$\Updownarrow$$

$$\Box\varphi \vdash \varphi.$$

- R is trans iff $R \circ R \subseteq R$ iff $\Box_R \leq \Box_R \circ \Box_R$

$$\Updownarrow$$

$$\Box\varphi \vdash \Box\Box\varphi.$$

Categories of Kripke Frames

$$(X, R_X) \xrightarrow{f} (Y, R_Y)$$

monotone map
preserves relations:

$w R_X v \Rightarrow f(w) R_Y f(v)$.

(forth)

also (+back) condition

$\boxed{\mathrm{Kr}}$ $\boxed{\mathrm{Kr}_B}$

$$\mathrm{Rel} \overset{\text{dual}}{\Longleftrightarrow} \mathrm{CABA}_s$$

$X \longmapsto \mathcal{P}(X)$

$R \longmapsto \Diamond_R$

$\Downarrow$

$$\mathrm{Kr} \overset{\text{dual}}{\Longleftrightarrow} \mathrm{CABAO}_s$$

$(X, R) \longmapsto (\mathcal{P}(X), \Diamond_R)$

Kr is topological over sets !

Fact 3. **Kr** is "topological over **Sets**",[15] meaning, concretely, the following. Given any family of functions $f_i : X \to Y_i$ $(i \in I)$ to Kripke frames (Y_i, R_i), the relation

$$wR_Xv \iff f_i(w)R_if_i(v) \text{ for all } i \in I, \quad \text{i.e.} \quad R_X = \bigcap_{i \in I}(f_i{}^{\dagger} \circ R_i \circ f_i),$$

is the (unique) "initial lift" of $\{f_i\}_{i \in I}$, i.e. the relation on X such that, given any function $g : Z \to X$, all $f_i \circ g$ are monotone from a frame (Z, R_Z) iff g is.

(In fact, Fact 3 holds of **Kr** in general, again with R^{α} in place of R_X.) One may note that the relation

$$\begin{tikzcd} & & (X, \underline{?}) \arrow[dll, "f_1"'] \arrow[d, "f_n"] \\ (Y_1, R_1) & \cdots & (Y_n, R_n) \end{tikzcd}$$

$\underline{R_X}$

the largest/coarsest relation preserved by all f_i.

Moreover,

- Initial lifts preserve refl, trans, symm. ..

  $\Rightarrow$ Subcats like Preord, Equiv are initially closed.

  $\Rightarrow$ They are also topological over Sets.

inclusion functors have left adjoints : e.g.

$$\begin{tikzcd} \mathrm{Kr} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathrm{Preord} \arrow[l, bend right, "i"'] \end{tikzcd}$$

$$(X, R) \longmapsto (X, R^*)$$

Also final lifts:

e.g.

$$\begin{tikzcd}
 & (X, \underline{?}) & \\
(X, R_1) \arrow[ur, "1_X"] & \cdots & (X, R_n) \arrow[ul, "1_X"']
\end{tikzcd}$$

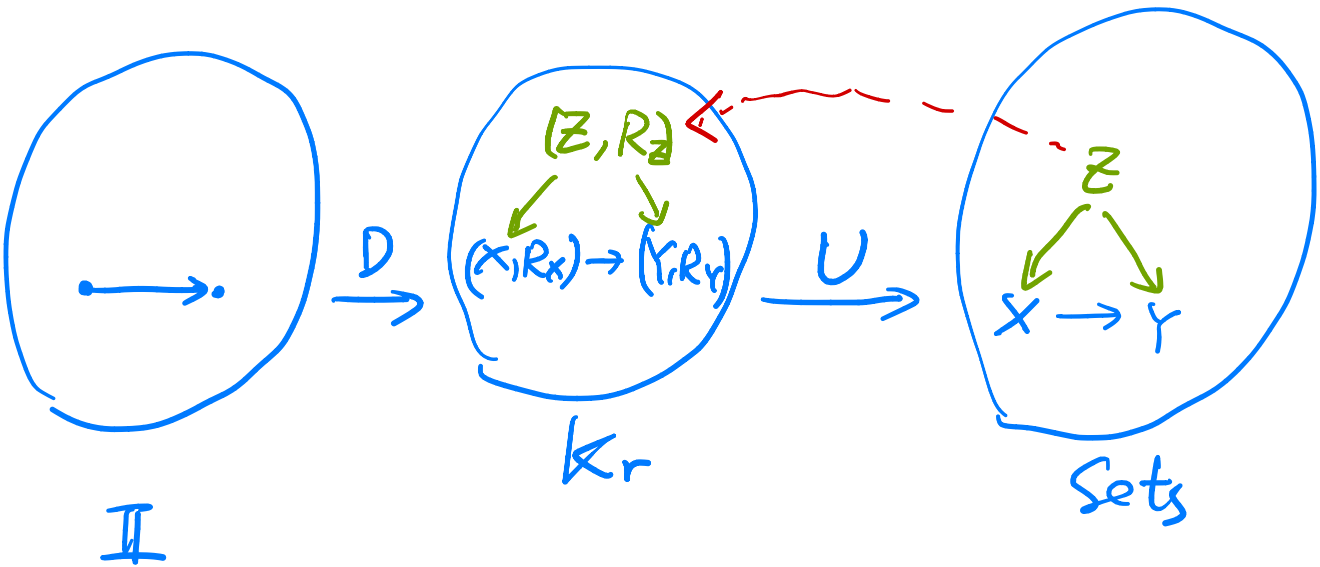
$R_X = (\bigvee_\alpha R_\alpha)^*$

In Preord

So common knowledge is no ad hoc construction!

We only need to look at (co)limits in Sets!

Another consequence, more relevant to this article, is that the forgetful functor U : **Kr** $\to$ **Sets** to the complete and cocomplete category **Sets** lifts limits and colimits—meaning that, given any (small) diagram D in **Kr**, its (co)limit exists on the (co)limit of $U \circ D$ in **Sets**. Most notably,



Examples.

$$\prod_{i\in I}(Y_i) = (X, \underline{?})$$

$$R_X := \bigcap_{i\in I}\left(p_i^{+} \circ R_i \circ p_i\right)$$

the largest/coarsest relation preserved by all f_i.

$$\begin{tikzcd}
 & (X, \underline{?}) \arrow[dl, "p_1"'] \arrow[dr, "p_n"] & \\
(Y_1, R_1) & \cdots & (Y_n, R_n)
\end{tikzcd}$$

$$\underbrace{(S, R_S)}_{\text{subframe on } S} \xrightarrow{\;i\;} (X, R_X)$$

Remarks:

These canonical maps are not in general bounded morphisms.

So we are working in Kr_B.

Semantics of DEL

PAL first:

unary operators:

$[\sigma!]$, $\langle \sigma! \rangle$

$[\sigma!]\varphi$

$\downarrow$

φ will be the case after σ is publicly and truthfully announced (observed)

A public announcement of σ

$$(X, R_X, [\![\ ]\!]_X)$$

$$S = [\![\sigma]\!]_X$$

$$(S, R_S) \xrightarrow{i} (X, R_X)$$

The reduction axioms follow!

$$[\![\, [\sigma!]\varphi \,]\!]_X = \forall_i [\![\varphi]\!]_S .$$

($w \in [\![\, [\sigma!]\varphi \,]\!]_X$ iff $v \in [\![\varphi]\!]_S$ whenever $v i w$)

$$[\![\sigma \Rightarrow \varphi]\!]_X = \forall_i \circ i^{-1} [\![\varphi]\!]_X .$$

At atoms level, we have

$$[\![p]\!]_S = i^{-1} [\![p]\!]_X$$

So we get the reduction axiom:

$$[\![\, [\sigma!] p \,]\!]_X = [\![\sigma \Rightarrow p]\!]_X$$

Another reduction axiom:

The dual of $R_s \circ i^t = i^t \circ R_x \circ i \circ i^t$

$$\Downarrow$$

$$\forall_i \circ \forall_{R_s^t} [\![\varphi]\!]_s = \forall_i \circ i^{-1} \circ \forall_{R_x^t} \circ \forall_i [\![\varphi]\!]_s$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$[\![[\sigma!]\Box\varphi]\!]_X \qquad\qquad\qquad [\![\sigma \Rightarrow \Box [\sigma!] \varphi]\!]_X$$

Remark

$[\sigma!] \sim \forall_i$ similar to $\Box \sim \forall_{R^t}$ but PAL (DEL) generalizes Kripke semantics by using $X \leftrightarrow Y$ between different Kripke models to interprete modal operators.

Event update:

Epistemic model: (X, R_X) with $[\![Pre(e)]\!]_X \subseteq X$ for $\forall e \in E$.

Event model: (E, R_E)

$$X \otimes E = \sum_{e \in E} [\![Pre(e)]\!]_X = \{ (w, e) \in X \times E \mid w \in [\![Pre(e)]\!]_X \}$$

coproduct injections

$$
\begin{tikzcd}
X \otimes E = \sum\limits_{e \in E} [\![\mathrm{Pre}(e)]\!]_X \arrow[r, hook, "i"] \arrow[rr, bend left, "p_E"] \arrow[dr, "p_X"] & X \times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
[\![\mathrm{Pre}(e)]\!]_X \arrow[u, tail, "q_e"] \arrow[r, hook, "i_e"'] \arrow[ur, phantom, "\urcorner", very near start] & X \arrow[u, tail, shift left, "q_{e'}"] &
\end{tikzcd}
$$

$R_{X \otimes E}$ is the initial lift of p_X and p_E.

Similar as before:

$$[\![[E,e]\varphi]\!]_X = \forall_{R_e^{+}} [\![\varphi]\!]_{X\otimes E}$$

$$[\![\langle E,e\rangle\varphi]\!]_X = \exists_{R_e^{+}} [\![\varphi]\!]_{X\otimes E}$$

and the reduction axioms follow due to some duality.

Some remarks:

We conclude this section with a remark on the significance of using the category **Kr**. We reviewed in this section that topological constructions (Subsection 2.4) and their canonical maps play essential rôles in the semantics of PAL and DEL. These constructions take place in **Kr** as opposed to the category $\mathbf{Kr_B}$, and the canonical maps are monotone maps of **Kr**, and not bounded morphisms of $\mathbf{Kr_B}$. Indeed, for

DEL to show interesting behaviors, the canonical maps—in particular, $p_X : X \otimes E \to X$, which amounts to $i : [\![\sigma]\!]_X \hookrightarrow X$ in the case of PAL—must not be bounded moprphisms. For, if p_X is a bounded morphism, then $[\![\varphi]\!]_{X\otimes E} = {p_X}^{-1}[\![\varphi]\!]_X$ for every φ and not just atomic p (this entails $[E,e]\varphi \equiv \mathrm{Pre}(e) \Rightarrow \varphi$ the same way as in (26))—this means that no event can teach agents anything. In other words, for events to teach agents something, they must bring about some change logically, and therefore the maps f representing them must not have logic-preserving duals f^{-1}.

Applications:

Integration with FOL (a sketch)

FOL + ML

(semantics in slice category Sets/X)

(Kripke semantics)

$\Downarrow$

FO-ML

(Kripke sheaves) → glues together local structures over worlds consistently

$\Downarrow$ + update

FO-DEL

(new sheaf semantics)

Potential future work:

what we need (or want) semantically.

- Our new application of the categorical methodology promises to be helpful on multiple fronts of the study of DEL. Naturally expected future work is to extend our approach to more vocabulary (e.g. common knowledge or μ-calculus), more types of logic (e.g. higher-order DEL or typed DEL), more structures (e.g. probability), and more general settings (e.g. intuitionistic or constructive modal logic).

- Use category theory to trace backward in the search for new and meaningful constructions.

  Kripke frame (X,R) to (X,R^*), where R^* is the reflexive and transitive closure of R.

  - One consequence of **Kr**, or a subcategory such as **Preord**, being topological over **Sets** is that it also has "final lifts", dual to initial lifts of Fact 3. E.g., given a family of preorders (X,R_α) $(\alpha \in A)$ on the same set X, such as "epistemic" relations R_α of agents $\alpha \in A$, consider an A-indexed family of identity maps $\{1_X\}_{\alpha \in A}$ in **Sets**; then its final lift in **Preord** comes with the epistemic relation for the "common knowledge" of the group A, i.e. $(\bigcup_\alpha R_\alpha)^*$.[17]

  Another consequence, more relevant to this article, is that the forgetful functor $U : \mathbf{Kr} \to \mathbf{Sets}$ to

  what about initial lift?

$$
\begin{tikzcd}
 & (X, ?) \arrow[dl] \arrow[d] \arrow[dr] & \\
(X, R_a) & (X, R_b) & (X, R_c)
\end{tikzcd}
$$

  - algebraic approach by Kurz and Palmigiano [31] uses ideas closely related to those in Section 3 of this article: They observe that the product update $X \otimes E$ is a subframe of the coproduct $X \times E = \sum_{e \in E} X$, and study the dual structure, i.e. a quotient of the product $\prod_{e \in E} \mathscr{P}(X)$.[31] Kurz and Palmigiano are well aware that these constructions do not take place in $\mathbf{Kr_B}$ or **CABAO** but rather in **Kr** and $\mathbf{CABAO_C}$. They stop

- Formalization of categorical semantics in Lean?
  - Giving a unified semantics for SMCDEL?